

# Characteristic classes of foliations via SAYD-twisted cocycles

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## Abstract

We have previously shown that the truncated Weil algebra of any Lie algebra is a Hopf-cyclic type complex with nontrivial coefficients. In this paper we apply this result to transfer the characteristic classes of transversely orientable foliations into the cyclic cohomology of the groupoid action algebra. Our result in codimension 1 matches with the only existing explicit computation done by Connes-Moscovici. In codimension 2 case, we carry out a constructive and explicit computation, by which we present the transverse fundamental class, the Godbillon-Vey class, and the other four residual classes as cyclic cocycles on the groupoid action algebra. The main object in charge in this new characteristic map is a SAYD-twisted cyclic cocycle of the same degree as the codimension. We construct such a cocycle by introducing an equivariant Hopf-cyclic cohomology and an equivariant cup product.

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# 1 Introduction

Following Connes-Moscovici [2], let  $\mathcal{A}_\Gamma := C_c^\infty(F^+) \rtimes \Gamma$ . Here  $F^+$  is the oriented frame bundle over  $\mathbb{R}^n$ , and  $\Gamma$  is a discrete subgroup of  $\text{Diff}^+(\mathbb{R}^n)$ , the group of orientation preserving diffeomorphisms of  $\mathbb{R}^n$ .

For an arbitrary  $\Gamma$ , the cyclic cohomology of  $\mathcal{A}_\Gamma$  is not known [1, Sect. III.2]. However, the Gelfand-Fuks cohomology of  $\mathfrak{a}_n$ , the Lie algebra of formal vector fields on  $\mathbb{R}^n$ , is finite dimensional and is embedded in this cohomology as a direct summand. In other words, there is a map, even in the level of complexes, which is a composite of two complicated maps:

$$\begin{array}{ccc}
 H_{\text{GF}}(\mathfrak{a}_n, \mathbb{C}) & \xrightarrow{\Phi \circ \mathcal{V}} & HP(\mathcal{A}_\Gamma) \\
 & \searrow \mathcal{V}_{\text{vanEst}} \quad \nearrow \Phi_{\text{Connes}} & \\
 & H_\tau(F^+, \mathbb{C}). &
 \end{array} \tag{1.1}$$

The first map is a van Est type map [2], which lands in the twisted cohomology computed by the Bott bicomplex [1, Prop. III.2.11], while the second map is due to Connes [1, Thm. III.2.14]. The representatives of the Gelfand-Fuks cohomology classes in  $H(\mathfrak{a}_n, \mathbb{C})$  are known thanks to the Vey basis of the cohomology of the truncated Weil algebra [4]. However it is difficult to transfer them to the cyclic cohomology of  $\mathcal{A}_\Gamma$ . The reader is referred to [3] for a complete account of the computation in codimension 1.

The Hopf-cyclic cohomology, invented by Connes-Moscovici for computing a local index formula [2], made it possible to have another

characteristic map

$$\chi_\tau : HP(\mathcal{H}_n, \mathbb{C}_\delta) \rightarrow HP(\mathcal{A}_\Gamma). \quad (1.2)$$

Here  $\mathcal{H}_n$  is the Connes-Moscovici Hopf algebra of codimension  $n$  and  $\mathbb{C}_\delta$  is the canonical one dimensional SAYD module over  $\mathcal{H}_n$ . One of the interesting features of this characteristic map is its simple presentation on the level of complexes,

$$\chi_\tau(h_1 \otimes \cdots \otimes h_n)(a^0, \dots, a^n) = \tau(a^0 h_1(a^1) \cdots h_n(a^n)). \quad (1.3)$$

Here  $\tau$  is the canonical trace on  $\mathcal{A}_\Gamma$  defined by

$$\tau(fU_\psi^*) = \begin{cases} \int_{F^+} f \varpi, & \text{if } \psi = \text{Id}, \\ 0, & \text{otherwise.} \end{cases} \quad (1.4)$$

It is also proved that  $HP(\mathcal{H}_n, \mathbb{C}_\delta)$  and  $H_{\text{GF}}(\mathfrak{a}_n, \mathbb{C})$  are canonically isomorphic, although once again this isomorphism is not easy to present [2, 11]. In view of (1.2), the only obstacle to transfer the characteristic classes of transversely orientable foliations to the cyclic cohomology of  $\mathcal{A}_\Gamma$  is a basis of the representatives of the Hopf-cyclic cohomology classes of  $\mathcal{H}_n$ . There is an intensive ongoing study [9, 10, 11] to investigate the Hopf-cyclic cohomology of the geometric bicrossed product Hopf algebras such as  $\mathcal{H}_n$ . The main idea is to use the bicrossed product construction to find the smallest complex by which one can compute the cohomology classes. This complex is found in [11] and is shown to be the codomain of the van Est isomorphism [11]. However, the return map from that complex to the Hopf-cyclic complex of the Hopf algebra is still missing.

In this paper we develop a new characteristic map, whose source is the Hopf-cyclic cohomology of  $\mathcal{K} := U(g\ell_n)$ , the enveloping algebra of the general linear Lie algebra  $g\ell_n$ . The Hopf algebra  $\mathcal{K}$  obviously is not as sophisticated as  $\mathcal{H}_n$ . Therefore to obtain and transfer the same classes, by considering the conservation of work, we would expect a characteristic map and a SAYD module more sophisticated than  $\chi_\tau$  and  $\mathbb{C}_\delta$  respectively.

In fact, the first step of our mission was taken in [13], where the authors showed that the truncated Weil algebra is a Hopf-cyclic complex. As a result, the characteristic classes of transversely orientable foliations can be calculated from  $HC(\mathcal{K}, V)$ . Here  $V := S(g\ell_n^*)_{[2n]}$ , the

algebra of  $n$ -truncated polynomials on  $g\ell_n$ , is a canonical and non-trivial SAYD module over  $\mathcal{K}$ .

The other important piece of this new characteristic map is a SAYD twisted cyclic  $n$ -cocycle  $\varphi \in C_{\mathcal{K}}^n(\mathcal{A}_{\Gamma}, V)$ . Next, we apply the cup product in Hopf cyclic cohomology introduced in [8] by Khalkhali and the first named author. In fact, since  $\varphi$  is a cyclic cocycle, we use the explicit formula derived in [12] to compute the characteristic classes of foliations as cyclic cocycles in  $HC(\mathcal{A}_{\Gamma})$  via

$$\chi_{\varphi} : HC^{\bullet}(\mathcal{K}, V) \rightarrow HC^{\bullet+n}(\mathcal{A}_{\Gamma}), \quad \chi_{\varphi}(x) = x \cup \varphi. \quad (1.5)$$

In order to test our method we first carry out the computation for codimension 1 and observe that our result matches with the classes obtained by Connes-Moscovici in [3]. The result of [11] shows that the amount of work in codimension 2 is not comparable with that of codimension 1. However, we completely determine the representatives of all classes in  $HC(\mathcal{K}, V)$  in addition to an explicit formula for  $\varphi \in HC_{\mathcal{K}}^2(\mathcal{A}_{\Gamma}, V)$ . Then (1.5) yields our desired cyclic cocycles in  $HC(\mathcal{A}_{\Gamma})$ .

Throughout the paper, all vector spaces and their tensor products are over  $\mathbb{C}$  unless otherwise is specified. We use the Sweedler's notation for comultiplication and coaction. We denote the comultiplication of a coalgebra  $C$  by  $\Delta : C \rightarrow C \otimes C$  and its action on  $c \in C$  by  $\Delta(c) = c_{(1)} \otimes c_{(2)}$ . The image of  $v \in V$  under a left coaction  $\nabla : V \rightarrow C \otimes V$  is denoted by  $\nabla(v) = v_{<-1>} \otimes v_{<0>}$ , summation suppressed. By the coassociativity, we simply write  $\Delta(c_{(1)}) \otimes c_{(2)} = c_{(1)} \otimes \Delta(c_{(2)}) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$ . Unless stated otherwise, a Lie algebra  $\mathfrak{g}$  is finite dimensional with a basis  $\{X_i \mid 1 \leq i \leq n\}$  and a dual basis  $\{\theta^i \mid 1 \leq i \leq n\}$ . In particular, for  $\mathfrak{g} = g\ell_n$  we use  $\{Y_i^j \mid 1 \leq i, j \leq n\}$  for a basis and  $\{\theta_j^i \mid 1 \leq i, j \leq n\}$  for a dual basis. We denote the Weil algebra of  $\mathfrak{g}$  by  $W(\mathfrak{g})$ , and  $W(\mathfrak{g})_{[2n]}$  stands for the  $n$ -truncated Weil algebra of  $\mathfrak{g}$ . We denote the Kronecker symbol by  $\delta_j^i$ . We also adopt the Einstein summation convention on the repeating indices unless otherwise is stated. Finally, for the sake of simplicity we use

$$B_{\sigma(1)} \otimes \cdots \otimes B_{\sigma(q)} := \sum_{\sigma \in S_q} \text{sgn}(\sigma) B_{\sigma(1)} \otimes \cdots \otimes B_{\sigma(q)}$$

for any set of objects  $\{B_1, \dots, B_q\}$ . Here  $S_q$  is the group of all permutations on  $q$  objects and  $\text{sgn}(\sigma)$  stands for the signature of  $\sigma$ .

## 2 Preliminaries

In this section we bring all material needed for the sequel sections. The definition of Hopf-cyclic cohomology and a brief account of the Connes-Moscovici characteristic map is provided in the first subsection. The basics of the cyclic cohomology of Lie algebras are recalled in the other subsection.

### 2.1 Hopf-cyclic cohomology with coefficients

Let  $H$  be a Hopf algebra equipped with a character  $\delta : H \rightarrow \mathbb{C}$ , *i.e.* an algebra map, and a group-like element  $\sigma \in H$ , *i.e.*  $\Delta(\sigma) = \sigma \otimes \sigma$  and  $\varepsilon(\sigma) = 1$ . The pair  $(\delta, \sigma)$  is called a modular pair in involution (MPI for short) if

$$\delta(\sigma) = 1, \quad \text{and} \quad S_\delta^2 = \text{Ad}_\sigma, \quad (2.1)$$

where  $\text{Ad}_\sigma(h) = \sigma h \sigma^{-1}$ , for any  $h \in H$ , and  $S_\delta$  is defined by

$$S_\delta(h) = \delta(h_{(1)}) S(h_{(2)}), \quad h \in H. \quad (2.2)$$

A vector space  $M$  is called a right-left stable-anti-Yetter-Drinfeld module (SAYD for short) over  $H$  if it is a right  $H$ -module, a left  $H$ -comodule, and

$$\nabla(m \cdot h) = S(h_{(3)}) m_{<-1>} h_{(1)} \otimes m_{<0>} \cdot h_{(2)}, \quad m_{<0>} \cdot m_{<-1>} = m, \quad (2.3)$$

for any  $v \in V$  and  $h \in H$ . Using  $\delta$  and  $\sigma$  one endows  ${}^\sigma\mathbb{C}_\delta := \mathbb{C}$ , the field of complex numbers, with a right module and left comodule structures over  $H$ . This way  ${}^\sigma\mathbb{C}_\delta$  is a SAYD module over the Hopf algebra  $H$  if and only if  $(\delta, \sigma)$  is an MPI.

Now let  $M$  be a right-left SAYD module over  $H$  and  $C$  an  $H$ -module coalgebra, that is,  $\Delta(h(c)) = h_{(1)}(c_{(1)}) \otimes h_{(2)}(c_{(2)})$  for any  $h \in H$  and  $c \in C$ . Then we have the graded space

$$C_H(C, M) := \bigoplus_{q \geq 0} C_H^q(C, M), \quad C_H^q(C, M) := M \otimes_H C^{\otimes q+1} \quad (2.4)$$

with the coface operators

$$\begin{aligned} \mathfrak{d}_i : C_H^q(C, M) &\rightarrow C_H^{q+1}(C, M), \quad 0 \leq i \leq q+1 \\ \mathfrak{d}_i(m \otimes_H c^0 \otimes \cdots \otimes c^q) &= m \otimes c^0 \otimes \cdots \otimes \Delta(c^i) \otimes \cdots \otimes c^q, \\ \mathfrak{d}_{q+1}(m \otimes_H c^0 \otimes \cdots \otimes c^q) &= \\ m_{<0>} \otimes_H c_{(2)}^0 \otimes c^1 \otimes \cdots \otimes c^q \otimes m_{<-1>}(c_{(1)}^0), \end{aligned} \quad (2.5)$$

the codegeneracy operators

$$\begin{aligned}\mathfrak{s}_j : C_H^q(C, M) &\rightarrow C_H^{q-1}(C, M), \quad 0 \leq j \leq q-1 \\ \mathfrak{s}_j(m \otimes_H c^0 \otimes \cdots \otimes h^q) &= m \otimes_H c^0 \otimes \cdots \otimes \varepsilon(c^{j+1}) \otimes \cdots \otimes c^q,\end{aligned}\quad (2.6)$$

and the cyclic operator

$$\begin{aligned}\mathfrak{t}_q : C_H^q(C, M) &\rightarrow C_H^q(C, M), \\ \mathfrak{t}_q(m \otimes_H c^0 \otimes \cdots \otimes c^q) &= m_{<0>} \otimes_H c^1 \otimes \cdots \otimes c^q \otimes m_{<-1>}(c^0).\end{aligned}\quad (2.7)$$

The graded space  $C_H(C, M)$  endowed with the above operators forms a cocyclic module. Using the above operators one defines the Hochschild coboundary  $b$  and the Connes boundary operator  $B$ ,

$$b : C_H^q(C, M) \rightarrow C_H^{q+1}(C, M), \quad b := \sum_{i=0}^{q+1} (-1)^i \mathfrak{d}_i, \quad (2.8)$$

$$B : C_H^q(C, M) \rightarrow C_H^{q-1}(C, M), \quad B := \left( \sum_{i=0}^q (-1)^{qi} \mathfrak{t}^i \right) \mathfrak{s}_{q-1} \mathfrak{t}. \quad (2.9)$$

The cyclic cohomology of  $C$  under the symmetry of  $H$  with coefficients in the SAYD module  $M$ , which is denoted by  $HC(C, M)$ , is defined to be the cyclic cohomology of the complex  $C_H(C, M)$ .

For  $C = H$ , the map

$$\begin{aligned}\mathcal{J} : M \otimes_H H^{\otimes(n+1)} &\rightarrow M \otimes H^{\otimes n}, \\ \mathcal{J}(m \otimes_H h^0 \otimes \cdots \otimes h^n) &= m h^0_{(1)} \otimes S(h_{(2)}) \cdot (h^1 \otimes \cdots \otimes h^n).\end{aligned}\quad (2.10)$$

identifies the standard Hopf-cyclic complex (2.4) with

$$C(H, M) := \bigoplus_{q \geq 0} C^q(H, M), \quad C^q(H, M) := M \otimes H^{\otimes q}. \quad (2.11)$$

Then the coface operators become

$$\begin{aligned}\mathfrak{d}_i : C^q(H, M) &\rightarrow C^{q+1}(H, M), \quad 0 \leq i \leq q+1 \\ \mathfrak{d}_0(m \otimes h^1 \otimes \cdots \otimes h^q) &= m \otimes 1 \otimes h^1 \otimes \cdots \otimes h^q, \\ \mathfrak{d}_i(m \otimes h^1 \otimes \cdots \otimes h^q) &= \\ m \otimes h^1 \otimes \cdots \otimes h^i_{(1)} \otimes h^i_{(2)} \otimes \cdots \otimes h^q, &\quad 1 \leq i \leq q \\ \mathfrak{d}_{q+1}(m \otimes h^1 \otimes \cdots \otimes h^q) &= m_{<0>} \otimes h^1 \otimes \cdots \otimes h^q \otimes m_{<-1>},\end{aligned}$$

the codegeneracy operators

$$\begin{aligned}\mathfrak{s}_j &: C^q(H, M) \rightarrow C^{q-1}(H, M), \quad 0 \leq j \leq q-1 \\ \mathfrak{s}_j(m \otimes h^1 \otimes \cdots \otimes h^q) &= m \otimes h^1 \otimes \cdots \otimes \varepsilon(h^{j+1}) \otimes \cdots \otimes h^q,\end{aligned}$$

and the cyclic operator becomes

$$\begin{aligned}\mathfrak{t} &: C^q(H, M) \rightarrow C^q(H, M), \\ \mathfrak{t}(m \otimes h^1 \otimes \cdots \otimes h^q) &= m_{<0>} h^1_{(1)} \otimes S(h^1_{(2)}) \cdot (h^2 \otimes \cdots \otimes h^q \otimes m_{<-1>}).\end{aligned}$$

Let  $A$  be a  $H$ -module algebra, that is, a (left)  $H$ -module and

$$h(ab) = h_{(1)}(a)h_{(2)}(b), \quad h(1_A) = \varepsilon(h)1_A, \quad \forall h \in H, a \in A.$$

One endows  $V \otimes A^{\otimes n+1}$  with the action of  $H$

$$(v \otimes a^0 \otimes \cdots \otimes a^q) \cdot h = mh_{(1)} \otimes S(h_{(q+2)})a^0 \otimes \cdots \otimes S(h_{(2)})a^q, \quad (2.12)$$

and forms

$$C_H^n(A, V) = \text{Hom}_H(V \otimes A^{\otimes n+1}, \mathbb{C}) \quad (2.13)$$

as the space of  $H$ -linear maps. It is checked in [5] that for any  $v \otimes \tilde{a} := v \otimes a^0 \otimes \cdots \otimes a^{n+2} \in V \otimes A^{\otimes n+2}$  the morphisms

$$\begin{aligned}(\partial_i \varphi)(v \otimes \tilde{a}) &= \varphi(v \otimes a^0 \otimes \cdots \otimes a^i a^{i+1} \otimes \cdots \otimes a^{n+1}), \quad 0 \leq i < n, \\ (\partial_{n+1} \varphi)(v \otimes \tilde{a}) &= \varphi(v_{<0>} \otimes (S^{-1}(v_{<-1>})a^{n+1})a^0 \otimes a^1 \otimes \cdots \otimes a^n), \\ (\sigma_i \varphi)(v \otimes \tilde{a}) &= \varphi(v \otimes a^0 \otimes \cdots \otimes a^i 1 \otimes \cdots \otimes a^{n-1}), \quad 0 \leq i \leq n-1, \\ (\tau \varphi)(v \otimes \tilde{a}) &= \varphi(v_{<0>} \otimes (S^{-1}(v_{<-1>})a^n) \otimes a^0 \otimes \cdots \otimes a^{n-1})\end{aligned}$$

define a cocyclic module structure on  $C_H^n(A, V)$ , whose cyclic cohomology is denoted by  $HC_H(A, V)$ .

One uses  $HC(H, V)$  and  $HC_H(A, V)$  to define a cup product

$$HC_H^p(A, V) \otimes HC_H^q(H, V) \rightarrow HC^{p+q}(A),$$

whose definition can be found in [12, 8].

As the simplest example, one notes that the cup product with the 0-cocycle  $\tau \in C_H^0(A, {}^\sigma \mathbb{C}_\delta)$  defines the Connes-Moscovici characteristic map [2, 3],

$$\begin{aligned}\chi_\tau &: HC^\bullet(H, {}^\sigma \mathbb{C}_\delta) \rightarrow HC^\bullet(A) \\ \chi_\tau(h^1 \otimes \cdots \otimes h^n)(a^0 \otimes \cdots \otimes a^n) &= \tau(a^0 h^1(a^1) \cdots h^n(a^n)).\end{aligned} \quad (2.14)$$

## 2.2 Lie algebra (co)homology

In this subsection, after recalling the Lie algebra (co)homology, we summarize our work in [13] on the cyclic cohomology of Lie algebras with coefficients in SAYD modules.

Let  $\mathfrak{g}$  be a Lie algebra and  $V$  be a right  $\mathfrak{g}$ -module. The Lie algebra homology complex is defined to be

$$C(\mathfrak{g}, V) = \bigoplus_{q \geq 0} C_q(\mathfrak{g}, V), \quad C_q(\mathfrak{g}, V) := \wedge^q \mathfrak{g} \otimes V \quad (2.15)$$

with the Chevalley-Eilenberg boundary map

$$\cdots \xrightarrow{\partial_{\text{CE}}} C_2(\mathfrak{g}, V) \xrightarrow{\partial_{\text{CE}}} C_1(\mathfrak{g}, V) \xrightarrow{\partial_{\text{CE}}} V, \quad (2.16)$$

$$\begin{aligned} \partial_{\text{CE}}(X_0 \wedge \cdots \wedge X_{q-1} \otimes v) &= \sum_{i=0}^{q-1} (-1)^i X_0 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge X_{q-1} \otimes v \cdot X_i + \\ &\sum_{0 \leq i < j \leq q-1} (-1)^{i+j} [X_i, X_j] \wedge X_0 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge \widehat{X}_j \wedge \cdots \wedge X_{q-1} \otimes v. \end{aligned}$$

The homology of the complex  $(C(\mathfrak{g}, V), \partial_{\text{CE}})$  is called the Lie algebra homology of  $\mathfrak{g}$  with coefficients in  $V$  and it is denoted by  $H_\bullet(\mathfrak{g}, V)$ . In a dual fashion one defines the Lie algebra cohomology complex

$$W(\mathfrak{g}, V) = \bigoplus_{q \geq 0} W^q(\mathfrak{g}, V), \quad W^q(\mathfrak{g}, V) = \text{Hom}(\wedge^q \mathfrak{g}, V), \quad (2.17)$$

where  $\text{Hom}(\wedge^q \mathfrak{g}, V)$  is the vector space of all alternating linear maps on  $\mathfrak{g}^{\otimes q}$  with values in  $V$ . The Chevalley-Eilenberg coboundary

$$V \xrightarrow{d_{\text{CE}}} W^1(\mathfrak{g}, V) \xrightarrow{d_{\text{CE}}} W^2(\mathfrak{g}, V) \xrightarrow{d_{\text{CE}}} \cdots, \quad (2.18)$$

is defined by

$$\begin{aligned} d_{\text{CE}}(\alpha)(X_0, \dots, X_q) &= \sum_{0 \leq i < j \leq q} (-1)^{i+j} \alpha([X_i, X_j], X_0 \dots \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_q) + \\ &\sum_{i=0}^q (-1)^{i+1} \alpha(X_0, \dots, \widehat{X}_i, \dots, X_q) \cdot X_i. \end{aligned} \quad (2.19)$$



Alternatively, we may identify  $W^q(\mathfrak{g}, V)$  with  $\wedge^q \mathfrak{g}^* \otimes V$  and the coboundary  $d_{\text{CE}}$  with

$$\begin{aligned} d_{\text{CE}}(v) &= -\theta^i \otimes v \cdot X_i, & d_{\text{CE}}(\beta \otimes v) &= d_{\text{dR}}(\beta) \otimes v - \theta^i \wedge \beta \otimes v \cdot X_i, \\ d_{\text{dR}} : \wedge^p \mathfrak{g}^* &\rightarrow \wedge^{p+1} \mathfrak{g}^*, & d_{\text{dR}}(\theta^i) &= -\frac{1}{2} C_{jk}^i \theta^j \wedge \theta^k \end{aligned} \quad (2.20)$$

The cohomology of the complex  $(W(\mathfrak{g}, V), d_{\text{CE}})$ , the Lie algebra cohomology of  $\mathfrak{g}$  with coefficients in  $V$ , is denoted by  $H^\bullet(\mathfrak{g}, V)$ .

In this paper we are particularly interested in the SAYD modules over the universal enveloping algebra  $U(\mathfrak{g})$  of a Lie algebra  $\mathfrak{g}$ . By our results in [13], such SAYD modules are in fact obtained from the SAYD modules over the Lie algebra  $\mathfrak{g}$ .

**Definition 2.1** ([13]). *A vector space  $V$  is a left comodule over the Lie algebra  $\mathfrak{g}$  if there is a linear map*

$$\nabla_{\mathfrak{g}} : V \rightarrow \mathfrak{g} \otimes V, \quad \nabla_{\mathfrak{g}}(v) = v_{[-1]} \otimes v_{[0]} \quad (2.21)$$

such that

$$v_{[-2]} \wedge v_{[-1]} \otimes v_{[0]} = 0,$$

where

$$v_{[-2]} \otimes v_{[-1]} \otimes v_{[0]} = v_{[-1]} \otimes (v_{[0]})_{[-1]} \otimes (v_{[0]})_{[0]}.$$

It is clear that left  $\mathfrak{g}$ -comodules and right  $S(\mathfrak{g}^*)$ -modules are identical.

**Definition 2.2** ([13]). *Let  $V$  be a right module and left comodule over a Lie algebra  $\mathfrak{g}$ . We call  $V$  a right-left anti-Yetter-Drinfeld module (AYD module) over  $\mathfrak{g}$  if*

$$\nabla_{\mathfrak{g}}(v \cdot X) = v_{[-1]} \otimes v_{[0]} \cdot X + [v_{[-1]}, X] \otimes v_{[0]}. \quad (2.22)$$

Moreover,  $V$  is called stable if

$$v_{[0]} \cdot v_{[-1]} = 0. \quad (2.23)$$

Finally,  $V$  is said to be unimodular stable if  $V_{-\delta}$  is stable. Here the character  $\delta$  is defined by  $\delta = \text{Tr} \circ \text{ad} : \mathfrak{g} \rightarrow \mathbb{C}$  and  $V_{-\delta}$  is the deformation of  $V$  via

$$v \triangleleft X := v \cdot X - \delta(X)v.$$

**Example 2.3.** The truncated polynomial algebra  $V = S(\mathfrak{g}^*)_{[2n]}$ , of a Lie algebra  $\mathfrak{g}$ , is a unimodular SAYD module over  $\mathfrak{g}$  with the coadjoint action and the Koszul coaction defined by

$$\nabla_K : V \rightarrow \mathfrak{g} \otimes V, \quad \nabla_K(v) = \sum_{i=1}^n X_i \otimes v\theta^i. \quad (2.24)$$

Via the help of (unimodular) SAYD modules we generalize Lie algebra (co)homology complexes. Let us start with the Lie algebra homology by introducing the complex

$$C(\mathfrak{g}, V) = \bigoplus_{i \geq 0} \wedge^i \mathfrak{g} \otimes V, \quad \partial = \partial_{\text{CE}} + \partial_K \quad (2.25)$$

with the Chevalley-Eilenberg boundary and the Koszul coboundary

$$\partial_K : C_n(\mathfrak{g}, V) \rightarrow C_{n+1}(\mathfrak{g}, V), \quad \partial_K(e \otimes v) = v_{[-1]} \wedge e \otimes v_{[0]}. \quad (2.26)$$

Applying the Poincaré duality in the Lie algebra cohomology to the complex (2.25), see [13, Prop. 4.4], we obtain

$$W(\mathfrak{g}, V) = \bigoplus_{i \geq 0} \wedge^i \mathfrak{g}^* \otimes V \quad (2.27)$$

with  $d = d_{\text{CE}} + d_K$ , where  $d_{\text{CE}} : W^n(\mathfrak{g}, V) \rightarrow W^{n+1}(\mathfrak{g}, V)$  is the Chevalley-Eilenberg coboundary and

$$d_K : W^n(\mathfrak{g}, V) \rightarrow W^{n-1}(\mathfrak{g}, V), \quad d_K(\alpha \otimes v) = \iota(v_{[-1]})(\alpha) \otimes v_{[0]}.$$

Here  $\iota(X)$  is the contraction with respect to  $X$ .

In particular, we recover the (truncated) Weil algebra [13]:

$$W(\mathfrak{g}, S(\mathfrak{g}^*)) = W(\mathfrak{g}), \quad W(\mathfrak{g}, S(\mathfrak{g}^*)_{[2n]}) = W(\mathfrak{g})_{[2n]}. \quad (2.28)$$

### 3 SAYD-twisted cyclic cocycles

In this section we fix  $K$  to be a cocommutative Hopf subalgebra of a Hopf algebra  $H$ . Let  $A$  be an  $H$ -module algebra, and  $V$  be a SAYD module over  $K$ . We aim to develop a machinery to produce SAYD-twisted cyclic cocycles in  $HC_K(A, V)$ . In the first subsection we introduce equivariant Hopf-cyclic cohomology  $HC_K(H, V, N)$ , where  $N$  is a SAYD module over  $H$ . In the second subsection we construct a cup

product  $HC_K^p(H, V, N) \otimes HC_H^q(A, N) \rightarrow HC_K^{p+q}(A, V)$ . In the third and fourth subsection we apply the results of the first two subsections. This way we produce a nontrivial SAYD-twisted cyclic cocycle over the groupoid action algebra under the symmetry of the general linear Lie algebra with coefficients in the truncated polynomials on this Lie algebra.

### 3.1 Equivariant Hopf-cyclic cohomology

For a SAYD module  $N$  over  $H$  and a right module-left comodule  $V$  over  $K$  we define the graded space

$$C_K(H, V, N) = \bigoplus_{q \geq 0} C_K^q(H, V, N), \quad (3.1)$$

$$\mathcal{C}^q := C_K^q(H, V, N) := \text{Hom}_K(V, N \otimes_H H^{\otimes q+1}).$$

More precisely,  $\phi \in \mathcal{C}^q$  if and only if for any  $u \in K$  and any  $v \in V$

$$\phi(v \cdot u) = \phi(v) \cdot u, \quad (3.2)$$

where the right action of  $K$  on  $N \otimes_H H^{\otimes q+1}$  is the usual diagonal action, *i.e.*

$$(n \otimes_H h^0 \otimes \cdots \otimes h^q) \cdot u = n \otimes_H h^0 u_{(1)} \otimes \cdots \otimes h^q u_{(q+1)}. \quad (3.3)$$

For  $\phi \in C_K^q(H, V, N)$  and  $v \in V$ , we use the notation

$$\phi(v) = \phi(v)^{[-1]} \otimes_H \phi(v)^{[0]} \otimes \cdots \otimes \phi(v)^{[q]}. \quad (3.4)$$

Let us define the morphisms  $\mathfrak{d}_i : \mathcal{C}^q \rightarrow \mathcal{C}^{q+1}$ ,  $\mathfrak{s}_j : \mathcal{C}^q \rightarrow \mathcal{C}^{q-1}$ , and  $\mathfrak{t}_q : \mathcal{C}^q \rightarrow \mathcal{C}^q$  as

$$\begin{aligned} d_i(\phi)(v) &= \mathfrak{d}_i(\phi(v)), \quad 0 \leq i \leq q \\ d_{q+1}(\phi)(v) &= \mathfrak{d}_{q+1}(\phi(v_{<0>})) \triangleleft S(v_{<-1>}), \\ s_j(\phi)(v) &= \mathfrak{s}_j(\phi(v)), \quad 0 \leq j \leq q-1, \\ t_q(\phi)(v) &= \mathfrak{t}_q(\phi(v_{<0>})) \triangleleft S(v_{<-1>}), \end{aligned} \quad (3.5)$$

where the right action  $\triangleleft$  of  $K$  on  $N \otimes_H H^{\otimes q+1}$  is defined by

$$(n \otimes_H h^0 \otimes \cdots \otimes h^q) \triangleleft u = n \otimes_H h^0 \otimes \cdots \otimes h^{q-1} \otimes h^q u. \quad (3.6)$$

Here the morphisms  $(\mathfrak{d}_i, s_j, t)$  are the usual morphisms of the cocyclic module  $C_H(H, N)$  defined in (2.5), (2.6) and (2.7).

**Theorem 3.1.** *If  $V$  and  $N$  are SAYD modules over  $K$  and  $H$  respectively, then the morphisms  $d_i, s_j$  and  $t$  define a cocyclic module structure on  $C_K(H, V, N)$ .*

*Proof.* Let us prove that the morphisms  $d_i, s_j$ , and  $t$  are well-defined. Indeed, it suffices to check that  $t, d_0$ , and  $s_n$  are well-defined as the other morphisms are made of these three. For  $d_0$  and  $s_n$  the task is obvious as  $\Delta : H \rightarrow H \otimes H$  and  $\varepsilon : H \rightarrow \mathbb{C}$  are multiplicative respectively. Let us check that  $t$  is well-defined. We have

$$\begin{aligned}
t(\phi)(v \cdot y) &= \tau(\phi((v \cdot y)_{<0>})) \triangleleft S((v \cdot y)_{<-1>}) \\
&= \tau(\phi(v_{<0>} \cdot y_{(2)})) \triangleleft S((S(y_{(1)})v_{<-1>}y_{(3)})) \\
&= \tau(\phi(v_{<0>}) \cdot y_{(2)}) \triangleleft S(y_{(3)})S(v_{<-1>}y_{(1)}) \\
&= \tau(\phi(v)^{[-1]} \otimes_H \phi(v)^{[0]}y_{(2)} \otimes \cdots \otimes \phi(v)^{[q]}y_{(q+2)}) \triangleleft S(y_{(q+3)})S(v_{<-1>}y_{(1)}) \\
&= \phi(v)^{[-1]}_{<0>} \otimes_H \phi(v)^{[1]}y_{(3)} \otimes \cdots \\
&\quad \cdots \otimes \phi(v)^{[q]}y_{(q+2)} \otimes \phi(v)^{[-1]}_{<-1>} \phi(v)^{[0]}y_{(2)} S(y_{(q+3)})S(v_{<-1>}y_{(1)}) \\
&= \phi(v)^{[-1]}_{<0>} \otimes_H \phi(v)^{[1]}y_{(1)} \otimes \cdots \\
&\quad \cdots \otimes \phi(v)^{[q]}y_{(q)} \otimes \phi(v)^{[-1]}_{<-1>} \phi(v)^{[0]}S(v_{<-1>}y_{(q+1)}) \\
&= t(\phi(v)) \cdot y.
\end{aligned} \tag{3.7}$$

In the second and the sixth equalities we use the fact that  $K$  is co-commutative.

Let us next show that  $C_K(H, V, N)$  is a cocyclic module which means that  $d_i, s_j$  and  $t$  satisfy

$$d_j d_i = d_i d_{j-1}, \quad i < j, \quad s_j s_i = s_i s_{j+1}, \quad i \leq j \tag{3.8}$$

$$s_j d_i = \begin{cases} d_i s_{j-1} & i < j \\ \text{Id}_q & \text{if } i = j \text{ or } i = j + 1 \\ d_{i-1} s_j & i > j + 1; \end{cases} \tag{3.9}$$

$$t_{q+1} d_i = d_{i-1} t_q, \quad 1 \leq i \leq q + 1, \quad t_{q+1} d_0 = d_{q+1} \tag{3.10}$$

$$t_{q-1} s_i = s_{i-1} t_q, \quad 1 \leq i \leq q - 1, \quad t_q s_0 = s_{q-1} t_q^2 \tag{3.11}$$

$$t_q^{q+1} = \text{Id}_q. \tag{3.12}$$

The equalities (3.8), (3.9) and (3.11) follow directly from their counterparts for the operators  $\mathfrak{d}_i, \mathfrak{s}_j$  and  $\mathfrak{t}$ .

As for (3.10), we check the case  $i = q + 1$ . Indeed

$$\begin{aligned}
t_{q+1}(d_{q+1}(\phi))(v) &= \mathfrak{t}_{q+1}(d_{q+1}(\phi)(v_{<0>})) \triangleleft S((v_{<-1>})) \\
&= \mathfrak{t}_{q+1}(\mathfrak{d}_{q+1}(\phi(v_{<0> <0>})) \triangleleft S(v_{<0> <-1>})) \triangleleft S(v_{<-1>}) \\
&= \mathfrak{t}_{q+1}(\mathfrak{d}_{q+1}(\phi(v_{<0>})) \triangleleft S(v_{<-1>})) \triangleleft S(v_{<-2>}) \\
&= \mathfrak{t}_{q+1} \left( \phi(v_{<0>})^{[-1]}_{<0>} \otimes_H \phi(v_{<0>})^{[0]}_{(2)} \otimes \phi(v_{<0>})^{[1]} \otimes \cdots \otimes \right. \\
&\quad \left. \phi(v_{<0>})^{[q]} \otimes \phi(v_{<0>})^{[-1]}_{<-1>} \phi(v_{<0>})^{[0]}_{(1)} S(v_{<-1>}) \right) \triangleleft S(v_{<-2>}) \\
&= \phi(v_{<0>})^{[-1]}_{<0>} \otimes_H \phi(v_{<0>})^{[1]} \otimes \cdots \otimes \phi(v_{<0>})^{[q]} \otimes \\
&\quad \phi(v_{<0>})^{[-1]}_{<-2>} \phi(v_{<0>})^{[0]}_{(1)} S(v_{<-1>}) \otimes \\
&\quad \phi(v_{<0>})^{[-1]}_{<-1>} \phi(v_{<0>})^{[0]}_{(2)} S(v_{<-2>}) \\
&= \phi(v_{<0>})^{[-1]}_{<0>} \otimes_H \phi(v_{<0>})^{[1]} \otimes \cdots \otimes \phi(v_{<0>})^{[q]} \otimes \\
&\quad \Delta \left( \phi(v_{<0>})^{[-1]}_{<-1>} \phi(v_{<0>})^{[0]} S(v_{<-1>}) \right) \\
&= \mathfrak{d}_q \left( \left[ \phi(v_{<0>})^{[-1]}_{<0>} \otimes_H \phi(v_{<0>})^{[1]} \otimes \cdots \otimes \phi(v_{<0>})^{[q]} \otimes \right. \right. \\
&\quad \left. \left. \phi(v_{<0>})^{[-1]}_{<-1>} \phi(v_{<0>})^{[0]} \right] \triangleleft S(v_{<-1>}) \right) \\
&= \mathfrak{d}_q(\mathfrak{t}_q(\phi(v_{<0>})) \triangleleft S(v_{<-1>})) = d_q(t_q(\phi))(v).
\end{aligned} \tag{3.13}$$

A simple calculation yields one to

$$\begin{aligned}
t_q^{q+1}(\phi)(v) &= \phi(v_{<0>})^{[-1]}_{<0>} \otimes_H \phi(v_{<0>})^{[-1]}_{<-q-1>} \phi(v_{<0>})^{[0]} S(v_{<-1>}) \otimes \cdots \\
&\quad \cdots \otimes \phi(v_{<0>})^{[-1]}_{<-1>} \phi(v_{<0>})^{[q]} S(v_{<-q-1>}) \otimes \cdots \\
&= \phi(v_{<0>})^{[-1]}_{<0>} \phi(v_{<0>})^{[-1]}_{<-1>} \otimes_H \phi(v_{<0>})^{[0]} S(v_{<-1>}) \\
&\quad \cdots \otimes \phi(v_{<0>})^{[q]} S(v_{<-q-1>}) \\
&= \phi(v_{<0>})^{[-1]} \otimes_H \phi(v_{<0>})^{[0]} S(v_{<-1>}) \otimes \cdots \otimes \phi(v_{<0>})^{[q]} S(v_{<-q-1>}) \\
&= \phi(v_{<0>}) \cdot S(v_{<-1>}) = \phi(v_{<0>} \cdot S(v_{<-1>})) = \phi(v)
\end{aligned}$$

The last equality is held because of the fact that for any SAYD module  $V$  and any  $v \in V$ ,  $v_{<0>} \cdot S^{-1}(v_{<-1>}) = v$ , see [6, Lemma 4.9]. Here Since  $K$  is cocommutative  $S = S^{-1}$ .  $\square$

The cyclic cohomology of  $C_K(H, V, N)$  is denoted by  $HC_K(H, V, N)$ .

One notes that by taking  $K = \mathbb{C}$  and  $V = \mathbb{C}$  one recovers the usual Hopf-cyclic cohomology  $HC(H, N)$ .

### 3.2 Equivariant characteristic map

Let  $V$  and  $N$  be SAYD modules over  $K$  and  $H$  respectively. We define the map

$$\Psi : C_K^q(H, V, N) \otimes C_H^q(A, N) \longrightarrow C_K^q(A, V), \quad (3.14)$$

$$\begin{aligned} & \Psi(\phi \otimes \psi)(v \otimes a_0 \otimes \cdots \otimes a_q) \\ &= \psi(\phi(v)^{[-1]} \otimes \phi(v)^{[0]}(a_0) \otimes \phi(v)^{[1]}(a_1) \otimes \cdots \otimes \phi(v)^{[q]}(a_q)). \end{aligned} \quad (3.15)$$

One may check that  $\Psi$  is a map of cocyclic modules, where on the left hand side we consider the product of two cocyclic modules. This is enough to produce a generalization of the cup product in Hopf-cyclic cohomology [8, 12].

We define a new bicocyclic module by tensoring the cocyclic modules (2.7), and (2.13). The new bigraded module in the bidegree  $(p, q)$  is defined by

$$\mathcal{C}^{p,q} := \text{Hom}_K(V, N \otimes_H H^{\otimes p+1}) \otimes \text{Hom}_H(A^{\otimes q+1}, N) \quad (3.16)$$

with horizontal structure  $\vec{\partial}_i = \mathfrak{d}_i \otimes \text{Id}$ ,  $\vec{\sigma}_j = \mathfrak{s} \otimes \text{Id}$ , and  $\vec{\tau} = \mathfrak{t} \otimes \text{Id}$  and vertical structure  $\uparrow \partial_i = \text{Id} \otimes \partial_i$ ,  $\uparrow \sigma_j = \text{Id} \otimes \sigma_j$ , and  $\uparrow \tau = \text{Id} \otimes \tau$ . Obviously  $(\mathcal{C}^{\bullet,\bullet}, \vec{\partial}, \vec{\sigma}, \vec{\tau}, \uparrow \partial, \uparrow \sigma, \uparrow \tau)$  defines a bicocyclic module.

Now let us define the map

$$\Psi : \mathcal{D}^q \rightarrow \text{Hom}_K(V \otimes A^{\otimes q+1}, \mathbb{C}), \quad (3.17)$$

$$\begin{aligned} & \Psi(\phi \otimes \psi)(v \otimes a_0 \otimes \cdots \otimes a_q) \\ &= \psi(\phi(v)^{[-1]} \otimes \phi(v)^{[0]}(a_0) \otimes \phi(v)^{[1]}(a_1) \otimes \cdots \otimes \phi(v)^{[q]}(a_q)). \end{aligned}$$

Here  $\mathcal{D}^\bullet$  denotes the diagonal complex of the bicocyclic module  $\mathcal{C}^{\bullet,\bullet}$ . It is a cocyclic module whose  $q$ th component is  $\mathcal{C}^{q,q}$  and its cocyclic structure morphisms are  $\partial_i := \vec{\partial}_i \circ \uparrow \partial_i$ ,  $\sigma_j := \vec{\sigma}_j \circ \uparrow \sigma_j$ , and  $\tau := \vec{\tau} \circ \uparrow \tau$ .

**Proposition 3.2.** *The map  $\Psi$  is a well-defined map of cocyclic modules.*

*Proof.* Let us first show that  $\Psi$  is well-defined. Indeed, by using the fact that  $\phi$  is  $K$ -linear, we see

$$\begin{aligned}
& \Psi(\phi \otimes \psi)(vk_{(1)} \otimes S(k_{(q+2)})(a^0) \otimes \cdots \otimes S(k_{(2)})(a^q)) \\
&= \psi(\phi(vk_{(1)})^{[-1]} \otimes \phi(vk_{(1)})^{[0]} S(k_{(q+2)})(a^0) \otimes \cdots \otimes \phi(vk_{(1)})^{[q]} S(k_{(2)})(a^q)) \\
&= \psi(\phi(v)^{[-1]} \otimes \phi(v)^{[0]} k_{(1)} S(k_{(2q+2)})(a^0) \otimes \cdots \otimes \phi(v)^{[q]} k_{(q+1)} S(k_{(q+2)})(a^q)) \\
&= \varepsilon(k) \psi(\phi(v)^{[-1]} \otimes \phi(v)^{[0]}(a^0) \otimes \cdots \otimes \phi(v)^{[q]}(a^q)) \\
&= \varepsilon(k) \Psi(\phi \otimes \psi)(v \otimes a^0 \otimes \cdots \otimes a^q).
\end{aligned}$$

Next, we show that  $\Psi$  commutes with the cocyclic structure morphisms. To this end, we need only to show the commutativity of  $\Psi$  with zeroth coface, the last codegeneracy and the cyclic operator, because these operators generate all cocyclic structure morphisms. We check it only for the cyclic operators and leave the rest to the reader.

$$\begin{aligned}
& \tau(\Psi(\phi \otimes \psi))(v \otimes a^0 \otimes \cdots \otimes a^q) \\
&= \Psi(\phi \otimes \psi)(v_{<0>} \otimes S^{-1}(v_{<-1>})(a^q) \otimes a^0 \otimes \cdots \otimes a^{q-1}) \\
&= \psi(\phi(v_{<0>})^{[-1]} \otimes \phi(v_{<0>})^{[0]} S^{-1}(v_{<-1>})(a^q) \otimes \phi(v_{<0>})^{[1]}(a^1) \otimes \cdots \\
&\quad \cdots \otimes \phi(v_{<0>})^{[q]}(a^q)).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
& \Psi(\mathbf{t}\phi \otimes \tau\psi)(v \otimes a^0 \otimes \cdots \otimes a^q) \\
&= \tau\psi(\mathbf{t}\phi(v)^{[-1]} \otimes \mathbf{t}\phi(v)^{[0]}(a^0) \otimes \cdots \otimes \mathbf{t}\phi(v)^{[q]}(a^q)) \\
&= \tau\psi(\mathbf{t}\phi(v)^{[-1]} \otimes \mathbf{t}\phi(v)^{[0]}(a^0) \otimes \cdots \otimes \mathbf{t}\phi(v)^{[q]}(a^q)) \\
&= \tau\psi(\phi(v_{<0>})^{[-1]}_{<0>} \otimes \phi(v)^{[1]}(a^0) \otimes \cdots \\
&\quad \cdots \otimes \phi(v)^{[q]}(a^{q-1}) \otimes \phi(v_{<0>})^{[-1]}_{<-1>} \phi(v_{<0>})^{[0]} S(v_{<-1>})(a^q)) \\
&= \psi(\phi(v_{<0>})^{[-1]}_{<0>} \otimes S^{-1}(\phi(v_{<0>})^{[-1]}_{<-1>}) \phi(v_{<0>})^{[-1]}_{<-2>} \phi(v_{<0>})^{[0]} S(v_{<-1>})(a^q) \otimes \\
&\quad \phi(v)^{[1]}(a^0) \otimes \cdots \otimes \phi(v)^{[q]}(a^{q-1})) \\
&= \psi(\phi(v_{<0>})^{[-1]} \otimes \phi(v_{<0>})^{[0]} S(v_{<-1>})(a^q) \otimes \phi(v_{<0>})^{[1]}(a^1) \otimes \cdots \otimes \phi(v_{<0>})^{[q]}(a^q)).
\end{aligned}$$

Since  $K$  is cocommutative  $S^2 = \text{Id}$ , and hence

$$\tau(\Psi(\phi \otimes \psi)) = \Psi(\mathbf{t}\phi \otimes \tau\psi).$$

□

**Theorem 3.3.** *Assume that  $K$  is a cocommutative Hopf subalgebra of a Hopf algebra  $H$ ,  $\mathcal{A}$  is a  $H$ -module algebra, and  $V$  and  $N$  are SAYD modules over  $K$  and  $H$  respectively. Then the map  $\Psi$  defines a cup product*

$$HC_K^p(H, V, N) \otimes HC_H^q(A, N) \rightarrow HC_K^{p+q}(A, V). \quad (3.18)$$

*Proof.* Let  $[\phi] \in HC_K^p(H, V, N)$  and  $[\psi] \in HC_H^q(A, N)$ . Without loss of generality we assume that  $\phi$  and  $\psi$  are respectively cyclic cocycles horizontally and vertically. This implies that  $\phi \otimes \psi$  is a  $(b, B)$  cocycle of degree  $p + q$  in total complex of  $\mathcal{C}^{\bullet, \bullet}$ . On the other hand by the cyclic Eilenberg-Zilber theorem [7], the total complex of  $\mathcal{C}^{\bullet, \bullet}$  is quasi-isomorphic with  $\mathcal{D}^\bullet$  via the Alexander-Witney map  $AW$ . So,  $AW(\phi \otimes \psi)$  is a  $(b, B)$  cocycle in  $\mathcal{D}^\bullet$ . Since  $\Psi$  is cyclic, we conclude that  $\Psi(AW(\phi \otimes \psi))$  defines a class in  $HC_K^{p+q}(A, V)$ .  $\square$

One notes that by setting  $K := \mathbb{C}$  as the trivial Hopf subalgebra and  $M = \mathbb{C}$  as the trivial SAYD module over  $K$  the above cup product becomes the cup product defined in [8, 12].

### 3.3 Equivariant charactrestic map for $\mathcal{H}_n$

In this subsection we apply our equivariant characteristic map we built in Subsection 3.2 to produce the desired cyclic cocycle on the groupoid action algebra.

Let us first recall the Connes-Moscovici Hopf algebra  $\mathcal{H}_n$  from [2, 3]. To this end let  $\mathfrak{h}_n$  be the Lie algebra generated by

$$\{X_k, Y_i^j, \delta_{jk\ell_1 \dots \ell_r}^i; i, j, k, \ell_1 \dots \ell_r = 1, \dots, n, r \in \mathbb{N}\} \quad (3.19)$$

with relations

$$\begin{aligned} [Y_i^j, Y_k^\ell] &= \delta_k^j Y_i^\ell - \delta_i^\ell Y_k^j, \quad [Y_i^j, X_k] = \delta_k^j X_i, \quad [X_k, X_\ell] = 0, \\ \delta_{jk\ell_1 \dots \ell_r}^i &= [X_{\ell_r}, \dots [X_{\ell_1}, \delta_{jk}^i] \dots], \quad [\delta_{jk\ell_1 \dots \ell_r}^i, \delta_{j'k'\ell'_1 \dots \ell'_r}^{i'}] = 0, \\ [Y_p^q, \delta_{j_1 j_2 j_3 \dots j_r}^i] &= \sum_{s=1}^r \delta_{j_s}^q \delta_{j_1 j_2 j_3 \dots j_{s-1} p j_{s+1} \dots j_r}^i - \delta_p^i \delta_{j_1 j_2 j_3 \dots j_r}^q, \\ \delta_{jk\ell_1 \dots \ell_r}^i &= \delta_{jk\ell_{\pi(1)} \dots \ell_{\pi(r)}}^i, \quad \forall \pi \in S_r. \end{aligned} \quad (3.20)$$

As an algebra,  $\mathcal{H}_n$  is  $U(\mathfrak{h}_n)$  modulo the (Bianchi-type) identities

$$\delta_{j\ell k}^i - \delta_{jk\ell}^i = \delta_{jk}^s \delta_{s\ell}^i - \delta_{j\ell}^s \delta_{sk}^i. \quad (3.21)$$



The coalgebra structure of  $\mathcal{H}_n$  is defined by a Leibniz rule that makes the groupoid action algebra  $\mathcal{A} = C_c^\infty(F^+) \rtimes \Gamma$  an  $\mathcal{H}_n$ -module algebra. In order to describe the action of  $\mathcal{H}_n$  explicitly, let us first identify  $F^+$  with  $\mathbb{R}^n \rtimes \mathrm{GL}^+(n, \mathbb{R})$  and use the local coordinates  $(x, y) \in F^+$ . A typical element of the groupoid action algebra is of the form  $fU_\phi^*$ , where  $U_\phi^*$  stands for  $\phi^{-1} \in \Gamma$  and  $f \in C_c^\infty(F^+)$ . The elements of  $\mathcal{H}_n$  act as the following operators.

$$\begin{aligned} X_k &= y_k^\mu \frac{\partial}{\partial x^\mu}, & X_k(fU_\phi^*) &:= X_k(f)U_\phi^*, \\ Y_i^j &= y_i^\mu \frac{\partial}{\partial y_\mu^j}, & Y_i^j(fU_\phi^*) &:= Y_i^j(f)U_\phi^*, \\ \delta_{jk\ell_1\ldots\ell_r}^i(fU_\phi^*) &= \gamma_{jk\ell_1\ldots\ell_r}^i(\phi) f U_\phi^*, \end{aligned} \quad (3.22)$$

where

$$\begin{aligned} \gamma_{jk\ell_1\ldots\ell_r}^i(\phi) &= X_{\ell_r} \cdots X_{\ell_1}(\gamma_{jk}^i(\phi)), \\ \gamma_{jk}^i(\phi)(x, y) &= (y^{-1} \cdot \phi'(x))^{-1} \cdot \partial_\mu \phi'(x) \cdot y)_j^i y_k^\mu. \end{aligned} \quad (3.23)$$

Therefore, for any  $a, b \in \mathcal{A}$  we have the Leibniz rule

$$\begin{aligned} Y_i^j(ab) &= Y_i^j(a)b + aY_i^j(b), \\ X_k(ab) &= X_k(a)b + aX_k(b) + \delta_{jk}^i(a)Y_i^j(b), \\ \delta_{jk}^i(ab) &= \delta_{jk}^i(a)b + a\delta_{jk}^i(b). \end{aligned} \quad (3.24)$$

Accordingly, we have

$$\Delta(Y_i^j) = Y_i^j \otimes 1 + 1 \otimes Y_i^j, \quad (3.25)$$

$$\Delta(\delta_{jk}^i) = \delta_{jk}^i \otimes 1 + 1 \otimes \delta_{jk}^i, \quad (3.26)$$

$$\Delta(X_k) = X_k \otimes 1 + 1 \otimes X_k + \delta_{jk}^i \otimes Y_i^j. \quad (3.27)$$

For simplicity, we will also employ the notation

$$\delta_{k\ell_1\ldots\ell_r}^a := \delta_{jk\ell_1\ldots\ell_r}^i, \quad Y_a := Y_i^j \quad a = \begin{pmatrix} i \\ j \end{pmatrix}. \quad (3.28)$$

Throughout this subsection we set  $\mathcal{H} := \mathcal{H}_n$ , the Connes-Moscovici Hopf algebra, and  $\mathcal{K} := U(\mathfrak{g}_0)$ , where  $\mathfrak{g}_0 := g\ell_n$ . We also let  $N = \mathbb{C}_\delta$  the SAYD module over  $\mathcal{H}$  where  $\delta : \mathcal{H} \rightarrow \mathbb{C}$  is the character defined on the generators by

$$\delta(Y_i^j) = \delta_i^j, \quad \delta(X_k) = \delta(\delta_{jk\ell_1\ldots\ell_r}^i) = 0, \quad 1 \leq i, j, k, l_t \leq n \quad (3.29)$$

and is extended on  $\mathcal{H}$  multiplicatively. Finally we set  $V = S(\mathfrak{g}_0^*)_{[2n]}$  with the canonical  $\mathcal{K}$ -SAYD module structure as recalled in (2.3).

Let  $\mathcal{A} := \mathcal{A}_\Gamma = C_c^\infty(F^+) \rtimes \Gamma$  and  $\tau$  be the canonical trace on  $\mathcal{A}$  defined in (1.4). Since  $\tau \in C_{\mathcal{H}}^0(\mathcal{A}, \mathbb{C}_\delta)$  and  $\tau$  is a cyclic cocycle [2], applying the cup product (3.18) we get the map of cocyclic modules

$$\begin{aligned} \chi_\tau^{\text{eq}} : HC_{\mathcal{K}}^q(\mathcal{H}, V, \mathbb{C}_\delta) &\longrightarrow HC_{\mathcal{K}}^q(\mathcal{A}, V) \\ \chi_\tau^{\text{eq}}(\phi)(v \otimes a_0 \otimes \cdots \otimes a_q) &= \tau \left( \phi(v)^{[0]}(a_0) \cdots \phi(v)^{[q]}(a_q) \right). \end{aligned} \quad (3.30)$$

We conclude this section by the identification of  $C_{\mathcal{K}}^q(\mathcal{K}, V, \mathbb{C}_\delta)$  with  $(V^* \otimes \mathbb{C}_\delta \otimes \mathcal{H}^{\otimes q})^{\mathfrak{g}_0}$ , where  $V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$  and  $\mathfrak{g}_0$  acts on  $V^* \otimes \mathcal{H}^{\otimes q}$  via

$$\begin{aligned} (\phi \otimes \mathbf{1} \otimes h^1 \otimes \cdots \otimes h^q)Z &= - \sum_{i=1}^q \phi \otimes \mathbf{1} \otimes h^1 \otimes \cdots \otimes \text{Ad}_Z(h^i) \otimes \cdots \otimes h^q \\ &+ \phi \otimes \delta(Z) \otimes h^1 \otimes \cdots \otimes h^q + \phi \cdot Z \otimes \mathbf{1} \otimes h^1 \otimes \cdots \otimes h^q. \end{aligned} \quad (3.31)$$

Here, the action of  $\mathfrak{g}_0$  on  $V^*$  is defined by  $(\phi \cdot Z)(v) = -\phi(v \cdot Z)$ .

The aforementioned identification is defined by the map

$$\begin{aligned} \mathcal{I} : (V^* \otimes \mathbb{C}_\delta \otimes \mathcal{H}^{\otimes q})^{\mathfrak{g}_0} &\rightarrow C_{\mathcal{K}}^q(\mathcal{H}, V, \mathbb{C}_\delta) \\ \mathcal{I}(\phi \otimes \mathbf{1} \otimes h^1 \otimes \cdots \otimes h^q)(v) &= \phi(v) \otimes_{\mathcal{H}} 1_{\mathcal{H}} \otimes h^1 \otimes \cdots \otimes h^q. \end{aligned} \quad (3.32)$$

**Proposition 3.4.** *The map  $\mathcal{I}$  defined in (3.32) is an isomorphism of vector spaces.*

*Proof.* Let us first check that  $\mathcal{I}$  is well-defined. Indeed,

$$\begin{aligned} \mathcal{I}(\phi \otimes \mathbf{1} \otimes h^1 \otimes \cdots \otimes h^q)(v \cdot Z) &= \phi(v \cdot Z) \otimes_{\mathcal{H}} 1_{\mathcal{H}} \otimes h^1 \otimes \cdots \otimes h^q \\ &= -(\phi \cdot Z)(v) \otimes_{\mathcal{H}} 1_{\mathcal{H}} \otimes h^1 \otimes \cdots \otimes h^q \\ &= - \sum_{i=1}^q \phi(v) \otimes_{\mathcal{H}} 1_{\mathcal{H}} \otimes h^1 \otimes \cdots \otimes \text{Ad}_Z(h^i) \otimes \cdots \otimes h^q \\ &+ \delta(Z)\phi(v) \otimes_{\mathcal{H}} 1_{\mathcal{H}} \otimes h^1 \otimes \cdots \otimes h^q \\ &= - \sum_{i=1}^q \phi(v) \otimes_{\mathcal{H}} 1_{\mathcal{H}} \otimes h^1 \otimes \cdots \otimes \text{Ad}_Z(h^i) \otimes \cdots \otimes h^q \\ &+ \phi(v) \otimes_{\mathcal{H}} Z \otimes h^1 \otimes \cdots \otimes h^q + \sum_{i=1}^q \phi(v) \otimes_{\mathcal{H}} 1_{\mathcal{H}} \otimes h^1 \otimes \cdots \otimes Z h^i \otimes \cdots \otimes h^q \\ &= (\phi(v) \otimes 1_{\mathcal{H}} \otimes h^1 \otimes \cdots \otimes h^q) \cdot Z = (\mathcal{I}(\phi \otimes \mathbf{1} \otimes h^1 \otimes \cdots \otimes h^q)(v)) \cdot Z. \end{aligned}$$

Next, we introduce an inverse map for  $\mathcal{I}$ . To this end we fix a basis for  $V$ , say  $\{v_1, \dots, v_m\}$ , with a dual basis  $\{\nu^1, \dots, \nu^m\}$  for  $V^*$ , and we define

$$\begin{aligned} \mathcal{I}^{-1} : C_{\mathcal{K}}^q(\mathcal{H}, V, \mathbb{C}_\delta) &\rightarrow (V^* \otimes \mathbb{C}_\delta \otimes \mathcal{H}^{\otimes q})^{\mathfrak{g}_0} \\ \mathcal{I}^{-1}(\phi) &= \sum_{i=1}^m \nu^i \otimes \phi(v_i)^{[-1]} \delta(\phi(v_i)^{[0]}_{(1)}) \otimes S(\phi(v_i)^{[0]}_{(2)}) \cdot \left( \phi(v_i)^{[1]} \otimes \dots \otimes \phi(v_i)^{[q]} \right). \end{aligned} \quad (3.33)$$

It is straightforward to check that  $\mathcal{I}^{-1}$  is independent of the choice of bases and is inverse to  $\mathcal{I}$ .  $\square$

### 3.4 A SAYD-twisted cyclic cocycle in codimension 1

In this subsection we keep the setting of Subsection 3.3 for  $n = 1$ . Our aim is to introduce an equivariant cyclic 1-cocycle  $\varphi \in C_{\mathcal{K}}^1(\mathcal{A}, V)$ . In order to apply (3.30), we shall consider the complex  $C_{\mathcal{K}}^1(\mathcal{H}, V, \mathbb{C}_\delta)$ , which in turn is identified with the  $\mathfrak{g}_0$ -invariant subspace  $(V^* \otimes \mathbb{C}_\delta \otimes \mathcal{H})^{\mathfrak{g}_0}$  via (3.32).

Let  $\{R\}$  be the basis for  $\mathfrak{g}_0^*$  as the dual basis of  $\{Y := Y_1^1\}$  for  $\mathfrak{g}_0$ . Let also  $\{1, R\}$  be the basis of  $V$  and  $\{1^*, S\}$  as the dual basis for  $V^*$ .

We define  $\varphi_0, \varphi_1 : V \otimes \mathcal{A}^{\otimes 2} \rightarrow \mathbb{C}$ , by

$$\varphi_0 = \chi_\tau^{\text{eq}}(1^* \otimes \mathbf{1} \otimes X), \quad \varphi_1 = \chi_\tau^{\text{eq}}(S \otimes \mathbf{1} \otimes \delta_1), \quad (3.34)$$

more precisely

$$\begin{aligned} \varphi_0((\alpha 1 + \beta \theta) \otimes a_0 \otimes a_1) &= \alpha \tau(a_0 X(a_1)), \\ \varphi_1((\alpha 1 + \beta \theta) \otimes a_0 \otimes a_1) &= \beta \tau(a_0 \delta_1(a_1)). \end{aligned} \quad (3.35)$$

**Lemma 3.5.** *The linear maps  $\varphi_0, \varphi_1 : V \otimes \mathcal{A}^{\otimes 2} \rightarrow \mathbb{C}$  defined in (3.35) are  $\mathcal{K}$ -equivariant.*

*Proof.* We have

$$\begin{aligned} \varphi_0((\alpha 1 + \beta \theta) \otimes Y(a_0) \otimes a_1) + \varphi_0((\alpha 1 + \beta \theta) \otimes a_0 \otimes Y(a_1)) &= \\ \alpha \tau(Y(a_0) X(a_1)) + \alpha \tau(a_0 X(Y(a_1))) &= \\ \alpha \tau(Y(a_0) X(a_1)) + \alpha \tau(a_0 Y(X(a_1))) - \alpha \tau(a_0 X(a_1)) &= \\ \alpha \tau(Y(a_0) X(a_1)) - \alpha \tau(Y(a_0) X(a_1)) &= 0, \end{aligned} \quad (3.36)$$

the third equality follows from the integration by parts property [2]

$$\tau(h(a)b) = \tau(aS_\delta(h)(b)), \quad \forall h \in \mathcal{H}, \forall a, b \in \mathcal{A}. \quad (3.37)$$

Similarly we have

$$\begin{aligned} & \varphi_1((\alpha 1 + \beta \theta) \otimes Y(a_0) \otimes a_1) + \varphi_1((\alpha 1 + \beta \theta) \otimes a_0 \otimes Y(a_1)) \\ &= \alpha \tau(Y(a_0)\delta_1(a_1)) + \alpha \tau(a_0\delta_1(Y(a_1))) \\ &= \alpha \tau(Y(a_0)\delta_1(a_1)) + \alpha \tau(a_0Y(\delta_1(a_1))) - \alpha \tau(a_0\delta_1(a_1)) \\ &= \alpha \tau(Y(a_0)\delta_1(a_1)) - \alpha \tau(Y(a_0)\delta_1(a_1)) = 0. \end{aligned} \quad (3.38)$$

□

**Lemma 3.6.** *The linear map  $\varphi_0 - \varphi_1 : V \otimes \mathcal{A}^{\otimes 2} \rightarrow \mathbb{C}$  is a Hochschild 1-cocycle.*

*Proof.* We have

$$\begin{aligned} & b(\varphi_0 - \varphi_1)((\alpha 1 + \beta \theta) \otimes a_0 \otimes a_1 \otimes a_2) = (\varphi_0 - \varphi_1)((\alpha 1 + \beta \theta) \otimes a_0 a_1 \otimes a_2) \\ & - (\varphi_0 - \varphi_1)((\alpha 1 + \beta \theta) \otimes a_0 \otimes a_1 a_2) + (\varphi_0 - \varphi_1)((\alpha 1 + \beta \theta) \otimes a_2 a_0 \otimes a_1) \\ & - (\varphi_0 - \varphi_1)(\alpha \theta \otimes Y(a_2) a_0 \otimes a_1) \\ &= \alpha \tau(a_0 a_1 X(a_2)) - \alpha \tau(a_0 X(a_1 a_2)) + \alpha \tau(a_2 a_0 X(a_1)) + \alpha \tau(Y(a_2) a_0 \delta_1(a_1)) \\ & - \beta \tau(a_0 a_1 \delta_1(a_2)) + \beta \tau(a_0 \delta_1(a_1 a_2)) - \beta \tau(a_2 a_0 \delta_1(a_1)) = 0. \end{aligned} \quad (3.39)$$

Here we have used (3.26) and (3.27). □

**Proposition 3.7.** *The Hochschild cocycle  $\varphi_0 - \varphi_1 : V \otimes \mathcal{A}^{\otimes 2} \rightarrow \mathbb{C}$ , defined in (3.35), is cyclic.*

*Proof.* By using the  $\delta$  invariancy of  $\tau$ , (3.26) and (3.27) we have

$$\begin{aligned} & t(\varphi_0 - \varphi_1)((\alpha 1 + \beta \theta) \otimes a_0 \otimes a_1) = (\varphi_0 - \varphi_1)((\alpha 1 + \beta \theta) \otimes a_1 \otimes a_0) \\ & - (\varphi_0 - \varphi_1)(\alpha \theta \otimes Y(a_1) \otimes a_0) \\ &= \alpha \tau(a_1 X(a_0)) + \alpha \tau(Y(a_1)\delta_1(a_0)) - \beta \tau(a_1 \delta_1(a_0)) \\ &= -\alpha \tau(a_0 X(a_1)) + \beta \tau(a_0 \delta_1(a_1)) = -(\varphi_0 - \varphi_1)((\alpha 1 + \beta \theta) \otimes a_0 \otimes a_1). \end{aligned}$$

□

### 3.5 A SAYD-twisted cyclic cocycle in codimension 2

Similar to the previous subsection, we keep the setting of Subsection 3.3 for  $n = 2$ . Our goal is to find a nontrivial cyclic 2-cocycle  $\phi \in C_{\mathcal{K}}^2(\mathcal{A}, V)$ .

Similar to the case  $n = 1$ , we apply (3.30) by considering  $C_{\mathcal{K}}^2(\mathcal{H}, V, \mathbb{C}_\delta) \cong (V^* \otimes \mathbb{C}_\delta \otimes \mathcal{H}^{\otimes 2})^{\mathfrak{g}_0}$ .

Let  $\{R_j^i \mid 1 \leq i, j \leq 2\}$  be the dual basis of  $g\ell_2$  with the pairing  $\langle Y_i^j, R_l^k \rangle = \delta_k^j \delta_l^i$ . We take

$$\left\{ 1, R_j^i, R_l^k R_q^p \mid \begin{pmatrix} k \\ l \end{pmatrix} \leq \begin{pmatrix} p \\ q \end{pmatrix} \right\}, \quad (3.40)$$

as a basis for  $V$  which is simplified by  $\{1, R^a, R^{ab} \mid a \leq b\}$ . The dual basis for  $V^*$  is expressed by  $\{1^*, S_a, S_{ab} \mid a \leq b\}$ .

We recall from [13] that the Koszul coaction (2.24) gives rise to a  $\mathcal{K}$ -coaction by the formula

$$\begin{aligned} \nabla_K : V &\rightarrow \mathcal{K} \otimes V, \\ \nabla_K(1) &= 1 \otimes 1 + Y_a \otimes R^a + \frac{1}{2!} Y_a Y_b \otimes R^{ab}, \\ \nabla_K(R^a) &= 1 \otimes R^a + Y_b \otimes R^{ab}, \\ \nabla_K(R^{ab}) &= 1 \otimes R^{ab}. \end{aligned} \quad (3.41)$$

We decompose  $V = V_0 \oplus V_1 \oplus V_2$ , where  $V_0 = \mathbb{C}\langle 1 \rangle$ ,  $V_1 = \mathbb{C}\langle R^a \rangle$ , and  $V_2 = \mathbb{C}\langle R^{ab} \rangle$ . Using this decomposition, any  $\psi \in \text{Hom}(V \otimes \mathcal{A}^{q+1}, \mathbb{C})$  is decomposed uniquely as  $\psi = \psi_0 + \psi_1 + \psi_2$  by  $\psi_i = \psi|_{V_i \otimes \mathcal{A}^{\otimes q+1}}$ .

We now consider the linear map  $\psi : V \otimes \mathcal{A}^{\otimes 3} \rightarrow \mathbb{C}$  with components

$$\psi_0 := \chi_\tau^{\text{eq}} \left( \gamma_1 1^* \otimes \mathbf{1} \otimes X_{\sigma(1)} \otimes X_{\sigma(2)} + \gamma_2 1^* \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^a \otimes X_{\sigma(2)} Y_a \right. \quad (3.42)$$

$$\left. + \gamma_3 1^* \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^a \delta_{\sigma(2)}^b Y_b \otimes Y_a + \gamma_4 1^* \otimes \mathbf{1} \otimes \delta_{\sigma(1)\sigma(2)}^a \otimes Y_a \right).$$

$$\begin{aligned} \psi_2 := \chi_\tau^{\text{eq}} \left( \alpha_1 S_{ab} \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^a \otimes \delta_{\sigma(2)}^b + \right. \quad (3.43) \\ \left. \alpha_2 S_{ab} \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^b \otimes \delta_{\sigma(2)}^a \right), \end{aligned}$$

$$\begin{aligned}
\psi_1 := & \chi_\tau^{\text{eq}} \left( \beta_1 S_a \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^a \otimes X_{\sigma(2)} + \beta_2 S_a \otimes \mathbf{1} \otimes X_{\sigma(1)} \otimes \delta_{\sigma(2)}^a \right. \\
& + \beta_3 S_a \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^a \delta_{\sigma(2)}^b \otimes Y_b + \beta_4 S_a \otimes \mathbf{1} \otimes Y_b \otimes \delta_{\sigma(1)}^a \delta_{\sigma(2)}^b \\
& + \beta_5 S_a \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^a Y_b \otimes \delta_{\sigma(2)}^b + \beta_6 S_a \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^b Y_b \otimes \delta_{\sigma(2)}^a \quad (3.44) \\
& + \beta_7 S_a \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^a \otimes \delta_{\sigma(2)}^b Y_b + \beta_8 S_a \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^b \otimes \delta_{\sigma(2)}^a Y_b \\
& \left. + \beta_9 S_a \otimes \mathbf{1} \otimes \Delta(\delta_{\sigma(1)\sigma(2)}^a) \right).
\end{aligned}$$

Our aim is to determine the coefficients  $\alpha_i, \beta_j, \gamma_k$ , such that  $\psi$  is a cyclic 2-cocycle. To do so we prove a series of technical lemmas.

**Lemma 3.8.** *The cochain  $\psi$  is  $\mathcal{K}$ -equivariant.*

*Proof.* We first simply observe that

$$\delta_{\sigma(1)}^j B_i \otimes B_{\sigma(2)} + \delta_{\sigma(2)}^j B_{\sigma(1)} \otimes B_i = \delta_i^j (B_{\sigma(1)} \otimes B_{\sigma(2)}). \quad (3.45)$$

Using (3.20), in view of the action of  $g\ell_2$  on the Hopf algebra  $\mathcal{H}$ , the equivariancy condition follows from

$$\begin{aligned}
& \text{ad}(Y_i^j) (\delta_{qr}^p \otimes Y_p^q) = \delta_r^j (\delta_{qi}^p \otimes Y_p^q), \\
& \text{ad}(Y_i^j) (\delta_{tr}^k \otimes \delta_{ln}^t) = \\
& \delta_l^j (\delta_{tr}^k \otimes \delta_{in}^t) - \delta_i^k (\delta_{tr}^j \otimes \delta_{ln}^t) + \delta_r^j (\delta_{ti}^k \otimes \delta_{ln}^t) + \delta_n^j (\delta_{tr}^k \otimes \delta_{li}^t).
\end{aligned}$$

□

We first observe that  $\psi \in C_{\mathcal{K}}^2(\mathcal{A}, V)$  is a Hochschild cocycle on  $V_2 \otimes \mathcal{A}^{\otimes n}$ .

**Lemma 3.9.** *For any  $\alpha_i, \beta_j, \gamma_k$ ,  $(b\psi)_2 = 0$ .*

*Proof.* The result follows directly from the application of the Hochschild coboundary map and the fact that  $\delta_k^a$  are derivations of  $\mathcal{A}$ . □

On the next move, we determine  $\alpha_i$ ,  $1 \leq i \leq 2$ , in such a way that  $\psi \in C_{\mathcal{K}}^2(\mathcal{A}, V)$  is a cyclic cocycle on  $V_2 \otimes \mathcal{A}^{\otimes n}$ .

**Lemma 3.10.** *For  $\alpha_1 = \alpha_2$ , we have  $(\tau\psi)_2 = \psi_2$ .*

*Proof.* By definition of the cyclic operator, we have

$$\tau\psi(R^{ab} \otimes a_0 \otimes a_1 \otimes a_2) = \psi_2(R^{ab} \otimes a_2 \otimes a_0 \otimes a_1). \quad (3.46)$$

Hence, by the integration by parts property (3.37),

$$\begin{aligned}
t\psi(R^{ab} \otimes a_0 \otimes a_1 \otimes a_2) &= \\
&\alpha_1 \tau \left( a_0 (-\delta^a_{\sigma(1)}) \delta^b_{\sigma(2)}(a_1) a_2 \right) + \alpha_1 \tau \left( a_0 \delta^b_{\sigma(2)}(a_1) (-\delta^a_{\sigma(1)})(a_2) \right) \\
&+ \alpha_2 \tau \left( a_0 (-\delta^b_{\sigma(1)}) \delta^a_{\sigma(2)}(a_1) a_2 \right) + \alpha_2 \tau \left( a_0 \delta^a_{\sigma(2)}(a_1) (-\delta^b_{\sigma(1)})(a_2) \right) \\
&= \alpha_1 \tau \left( a_0 \delta^b_{\sigma(1)}(a_1) \delta^a_{\sigma(2)}(a_2) \right) + \alpha_2 \tau \left( a_0 \delta^a_{\sigma(1)}(a_1) \delta^b_{\sigma(2)}(a_2) \right).
\end{aligned} \tag{3.47}$$

Therefore,  $(\tau\psi)_2 = \psi_2$  if and only if  $\alpha_1 = \alpha_2$ .  $\square$

As a result we set

$$\alpha_1 = \alpha_2 = r. \tag{3.48}$$

On the next step, we find a constraint on  $\beta_j$ 's such that  $\psi$  is a Hochschild cocycle over  $V_1 \otimes \mathcal{A}^{\otimes q+1}$ .

**Lemma 3.11.** *We have  $(b\psi)_1 = 0$  if and only if  $\{\beta_j \mid 1 \leq j \leq 9\}$  satisfy the system*

$$\begin{aligned}
\beta_1 - \beta_3 + \beta_7 + r &= 0 \\
\beta_3 + \beta_8 + r &= 0 \\
-\beta_2 - \beta_6 + \beta_8 &= 0 \\
\beta_4 - \beta_5 &= 0 \\
-\beta_4 - \beta_6 &= 0 \\
-\beta_5 + \beta_7 &= 0.
\end{aligned} \tag{3.49}$$

*Proof.* Let us remind the reader that we have to use the Koszul coaction (3.41) in the last coface operator.

$$\begin{aligned}
b\psi(R^a \otimes a_0 \otimes a_1 \otimes a_2 \otimes a_3) &= \\
\psi_1(R^a \otimes a_0 a_1 \otimes a_2 \otimes a_3) - \psi_1(R^a \otimes a_0 \otimes a_1 a_2 \otimes a_3) \\
+ \psi_1(R^a \otimes a_0 \otimes a_1 \otimes a_2 a_3) - \psi_1(R^a \otimes a_3 a_0 \otimes a_1 \otimes a_2) \\
+ \psi_2(R^{ab} \otimes Y_b(a_3) a_0 \otimes a_1 \otimes a_2).
\end{aligned} \tag{3.50}$$

Therefore, as a result of the tracial property [3, Thm. 6] and the

faithfullness [3, (3.12)] of the trace, we have  $(b\psi)_1 = 0$  if and only if

$$\begin{aligned}
& \beta_1 1 \otimes \delta^a_{\sigma(1)} \otimes \delta^b_{\sigma(2)} \otimes Y_b - \beta_2 1 \otimes \delta^b_{\sigma(1)} \otimes Y_b \otimes \delta^a_{\sigma(2)} \\
& - \beta_3 \left( 1 \otimes \delta^a_{\sigma(1)} \otimes \delta^b_{\sigma(2)} \otimes Y_b + 1 \otimes \delta^b_{\sigma(2)} \otimes \delta^a_{\sigma(1)} \otimes Y_b \right) \\
& + \beta_4 \left( 1 \otimes Y_b \otimes \delta^a_{\sigma(1)} \otimes \delta^b_{\sigma(2)} + 1 \otimes Y_b \otimes \delta^b_{\sigma(2)} \otimes \delta^a_{\sigma(1)} \right) \\
& - \beta_5 \left( 1 \otimes \delta^a_{\sigma(1)} \otimes Y_b \otimes \delta^b_{\sigma(2)} + 1 \otimes Y_b \otimes \delta^a_{\sigma(1)} \otimes \delta^b_{\sigma(2)} \right) \\
& - \beta_6 \left( 1 \otimes \delta^b_{\sigma(1)} \otimes Y_b \otimes \delta^a_{\sigma(2)} + 1 \otimes Y_b \otimes \delta^b_{\sigma(1)} \otimes \delta^a_{\sigma(2)} \right) \\
& + \beta_7 \left( 1 \otimes \delta^a_{\sigma(1)} \otimes \delta^b_{\sigma(2)} \otimes Y_b + 1 \otimes \delta^a_{\sigma(1)} \otimes Y_b \otimes \delta^b_{\sigma(2)} \right) \\
& + \beta_8 \left( 1 \otimes \delta^b_{\sigma(1)} \otimes \delta^a_{\sigma(2)} \otimes Y_b + 1 \otimes \delta^b_{\sigma(1)} \otimes Y_b \otimes \delta^a_{\sigma(2)} \right) \\
& + r 1 \otimes \delta^a_{\sigma(1)} \otimes \delta^b_{\sigma(2)} \otimes Y_b + r 1 \otimes \delta^b_{\sigma(1)} \otimes \delta^a_{\sigma(2)} \otimes Y_b = 0.
\end{aligned} \tag{3.51}$$

In other words,

$$\begin{aligned}
& (\beta_1 - \beta_3 + \beta_7 + k) 1 \otimes \delta^a_{\sigma(1)} \otimes \delta^b_{\sigma(2)} \otimes Y_b + (\beta_3 + \beta_8 + k) 1 \otimes \delta^b_{\sigma(1)} \otimes \delta^a_{\sigma(2)} \otimes Y_b \\
& + (-\beta_2 - \beta_6 + \beta_8) 1 \otimes \delta^b_{\sigma(1)} \otimes Y_b \otimes \delta^a_{\sigma(2)} + (\beta_4 - \beta_5) 1 \otimes Y_b \otimes \delta^a_{\sigma(1)} \otimes \delta^b_{\sigma(2)} \\
& + (-\beta_4 - \beta_6) 1 \otimes Y_b \otimes \delta^b_{\sigma(1)} \otimes \delta^a_{\sigma(2)} + (-\beta_5 + \beta_7) 1 \otimes \delta^a_{\sigma(1)} \otimes Y_b \otimes \delta^b_{\sigma(2)} = 0.
\end{aligned}$$

Accordingly we get the system (3.49).  $\square$

As a result we set  $\beta_j$ ,  $1 \leq j \leq 9$  to satisfy (3.49).

On the next step we determine  $\beta_j$ 's in such a way that  $\psi \in C_{\mathcal{K}}^2(\mathcal{A}, V)$  is a cyclic cocycle over  $(V_1 \oplus V_2) \otimes \mathcal{A}^{\otimes n}$ .

**Lemma 3.12.** *We have  $(\tau\psi)_1 = \psi_1$  if and only if  $\{\beta_j \mid 1 \leq j \leq 9\}$ , satisfy*

$$\begin{aligned}
& \beta_1 = \beta_2 = -r, \quad \beta_3 = \beta_4 = \beta_5 = -\beta_6 = \beta_7 = s, \\
& \beta_8 = -r - s, \quad \beta_9 = \frac{1}{2}r + s.
\end{aligned} \tag{3.52}$$

*Proof.* By the Koszul coaction, we have

$$t\psi(R^a \otimes a_0 \otimes a_1 \otimes a_2) = \psi_1(R^a \otimes a_2 \otimes a_0 \otimes a_1) - \psi_2(R^{ab} \otimes Y_b(a_2) \otimes a_0 \otimes a_1).$$



Accordingly,

$$\begin{aligned}
t\psi(R^a \otimes a_0 \otimes a_1 \otimes a_2) = & \\
& \beta_1 \tau \left( \delta^a_{\sigma(1)}(a_0) X_{\sigma(2)}(a_1) a_2 \right) + \beta_2 \tau \left( X_{\sigma(1)}(a_0) \delta^a_{\sigma(2)}(a_1) a_2 \right) \\
& + \beta_3 \tau \left( \delta^a_{\sigma(1)} \delta^b_{\sigma(2)}(a_0) Y_b(a_1) a_2 \right) + \beta_4 \tau \left( Y_b(a_0) \delta^a_{\sigma(1)} \delta^b_{\sigma(2)}(a_1) a_2 \right) \\
& + \beta_5 \tau \left( \delta^a_{\sigma(1)} Y_b(a_0) \delta^b_{\sigma(2)}(a_1) a_2 \right) + \beta_6 \tau \left( \delta^b_{\sigma(1)} Y_b(a_0) \delta^a_{\sigma(2)}(a_1) a_2 \right) \\
& + \beta_7 \tau \left( \delta^a_{\sigma(1)}(a_0) \delta^b_{\sigma(2)} Y_b(a_1) a_2 \right) + \beta_8 \tau \left( \delta^b_{\sigma(1)}(a_0) \delta^a_{\sigma(2)} Y_b(a_1) a_2 \right) \\
& + \beta_9 \tau \left( \Delta \left( \delta^a_{\sigma(1)\sigma(2)} \right) (a_0 \otimes a_1) a_2 \right) - r \tau \left( \delta^a_{\sigma(1)}(a_0) \delta^b_{\sigma(2)}(a_1) Y_b(a_2) \right) \\
& - r \tau \left( \delta^b_{\sigma(1)}(a_0) \delta^a_{\sigma(2)}(a_1) Y_b(a_2) \right).
\end{aligned} \tag{3.53}$$

Hence  $(t\psi)_1 = \psi_1$  if and only if

$$\begin{aligned}
& \beta_1 \left( -1 \otimes \delta^a_{\sigma(1)} X_{\sigma(2)} \otimes 1 - 1 \otimes X_{\sigma(2)} \otimes \delta^a_{\sigma(1)} \right) + \\
& \beta_2 \left( -1 \otimes X_{\sigma(1)} \delta^a_{\sigma(2)} \otimes 1 - 1 \otimes \delta^a_{\sigma(2)} \otimes X_{\sigma(1)} + 1 \otimes \delta^b_{\sigma(1)} \delta^a_{\sigma(2)} Y_b \otimes 1 \right. \\
& \quad \left. - 2 \cdot 1 \otimes \delta^a_{\sigma(1)\sigma(2)} \otimes 1 + 1 \otimes \delta^a_{\sigma(2)} \otimes \delta^b_{\sigma(1)} Y_b + 1 \otimes \delta^a_{\sigma(2)} Y_b \otimes \delta^b_{\sigma(1)} \right. \\
& \quad \left. - 1 \otimes \Delta \left( \delta^a_{\sigma(1)\sigma(2)} \right) \right) + \\
& \beta_3 \left( 1 \otimes \delta^a_{\sigma(1)} \delta^b_{\sigma(2)} Y_b \otimes 1 + 1 \otimes Y_b \otimes \delta^a_{\sigma(1)} \delta^b_{\sigma(2)} \right. \\
& \quad \left. + 1 \otimes \delta^a_{\sigma(1)} Y_b \otimes \delta^b_{\sigma(2)} + 1 \otimes \delta^b_{\sigma(2)} Y_b \otimes \delta^a_{\sigma(1)} \right) + \\
& \beta_4 \left( -1 \otimes \delta^a_{\sigma(1)} \delta^b_{\sigma(2)} Y_b \otimes 1 + 2 \cdot 1 \otimes \delta^a_{\sigma(1)\sigma(2)} \otimes 1 - 1 \otimes \delta^a_{\sigma(1)} \delta^b_{\sigma(2)} \otimes Y_b \right) + \\
& \beta_5 \left( 1 \otimes \delta^a_{\sigma(1)} \delta^b_{\sigma(2)} Y_b \otimes 1 - 2 \cdot 1 \otimes \delta^a_{\sigma(1)\sigma(2)} \otimes 1 + 1 \otimes \delta^a_{\sigma(1)} \delta^b_{\sigma(2)} \otimes Y_b + \right. \\
& \quad \left. + 1 \otimes \delta^b_{\sigma(2)} Y_b \otimes \delta^a_{\sigma(1)} + 1 \otimes \delta^b_{\sigma(2)} \otimes \delta^a_{\sigma(1)} Y_b + 1 \otimes \Delta \left( \delta^a_{\sigma(1)\sigma(2)} \right) \right) + \\
& \beta_6 \left( 1 \otimes \delta^b_{\sigma(1)} \delta^a_{\sigma(2)} Y_b \otimes 1 - 2 \cdot 1 \otimes \delta^a_{\sigma(1)\sigma(2)} \otimes 1 + 1 \otimes \delta^b_{\sigma(1)} \delta^a_{\sigma(2)} \otimes Y_b \right. \\
& \quad \left. + 1 \otimes \delta^a_{\sigma(2)} Y_b \otimes \delta^b_{\sigma(1)} - 1 \otimes \Delta \left( \delta^a_{\sigma(1)\sigma(2)} \right) + 1 \otimes \delta^a_{\sigma(2)} \otimes \delta^b_{\sigma(1)} Y_b \right) + \\
& \beta_7 \left( -1 \otimes \delta^a_{\sigma(1)} \delta^b_{\sigma(2)} Y_b \otimes 1 - 1 \otimes \delta^b_{\sigma(2)} Y_b \otimes \delta^a_{\sigma(1)} \right) + \\
& \beta_8 \left( -1 \otimes \delta^b_{\sigma(1)} \delta^a_{\sigma(2)} Y_b \otimes 1 - 1 \otimes \delta^a_{\sigma(2)} Y_b \otimes \delta^b_{\sigma(1)} \right) + \\
& \beta_9 \left( -2 \cdot 1 \otimes \delta^a_{\sigma(1)\sigma(2)} \otimes 1 - 1 \otimes \Delta \left( \delta^a_{\sigma(1)\sigma(2)} \right) \right) \\
& + r 1 \otimes \delta^b_{\sigma(2)} \otimes \delta^a_{\sigma(1)} Y_b + r 1 \otimes \delta^a_{\sigma(2)} \otimes \delta^b_{\sigma(1)} Y_b \\
& = \beta_1 1 \otimes \delta^a_{\sigma(1)} \otimes X_{\sigma(2)} + \beta_2 1 \otimes X_{\sigma(1)} \otimes \delta^a_{\sigma(2)} + \beta_3 1 \otimes \delta^a_{\sigma(1)} \delta^b_{\sigma(2)} \otimes Y_b \\
& \beta_4 1 \otimes Y_b \otimes \delta^a_{\sigma(1)} \delta^b_{\sigma(2)} + \beta_5 1 \otimes \delta^a_{\sigma(1)} Y_b \otimes \delta^b_{\sigma(2)} + \beta_6 1 \otimes \delta^b_{\sigma(1)} Y_b \otimes \delta^a_{\sigma(2)} \\
& + \beta_7 1 \otimes \delta^a_{\sigma(1)} \otimes \delta^b_{\sigma(2)} Y_b + \beta_8 1 \otimes \delta^b_{\sigma(1)} \otimes \delta^a_{\sigma(2)} Y_b + \beta_9 1 \otimes \Delta \left( \delta^a_{\sigma(1)\sigma(2)} \right).
\end{aligned}$$

Collecting the terms, we obtain the equations

$$\begin{aligned}
\beta_1 - \beta_2 &= 0 \\
\beta_2 - \beta_3 + \beta_4 - \beta_5 + \beta_6 + \beta_7 - \beta_8 &= 0 \\
\beta_1 - 2\beta_2 + 2\beta_4 - 2\beta_5 - 2\beta_6 - 2\beta_9 &= 0 \\
\beta_2 + \beta_6 + \beta_7 + k &= 0 \\
\beta_2 - \beta_3 + \beta_5 + \beta_6 - \beta_8 &= 0 \\
\beta_5 + \beta_8 + k &= 0 \\
\beta_3 + \beta_5 + \beta_6 - \beta_7 &= 0 \\
\beta_3 - \beta_4 &= 0 \\
-\beta_2 + \beta_5 - \beta_6 - 2\beta_9 &= 0.
\end{aligned} \tag{3.54}$$

Solving the systems (3.49) and (3.54) we obtain (3.52).  $\square$

As a result of Lemma 3.49 we set  $\beta_j$ ,  $1 \leq j \leq 9$  to satisfy (3.52).

Finally we determine  $\gamma_k$ ,  $1 \leq k \leq 4$  such that  $\psi \in C_{\mathcal{K}}^2(\mathcal{A}, V)$  is a Hochschild cocycle.

**Lemma 3.13.** *We have  $(b\psi)_0 = 0$  if and only if  $\{\gamma_k \mid 1 \leq k \leq 4\}$  satisfy*

$$\gamma_1 = \gamma_2 = r, \quad \gamma_3 = \gamma_4 = s. \tag{3.55}$$

*Proof.* Using the Koszul coaction (3.41), we have

$$\begin{aligned}
b\psi(1 \otimes a_0 \otimes a_1 \otimes a_2 \otimes a_3) &= \\
\psi_0(1 \otimes a_0 a_1 \otimes a_2 \otimes a_3) - \psi_0(1 \otimes a_0 \otimes a_1 a_2 \otimes a_3) \\
+ \psi_0(1 \otimes a_0 \otimes a_1 \otimes a_2 a_3) - \psi_0(1 \otimes a_3 a_0 \otimes a_1 \otimes a_2) \\
+ \psi_1(R^a \otimes Y_a(a_3)a_0 \otimes a_1 \otimes a_2) - \frac{1}{2!}\psi_2(R^{ab} \otimes Y_b Y_a(a_3)a_0 \otimes a_1 \otimes a_2).
\end{aligned}$$

As a result,  $(b\psi)_0 = 0$  if and only if

$$\begin{aligned}
&\gamma_1 \left( -1 \otimes \delta^a_{\sigma(1)} \otimes Y_a \otimes X_{\sigma(2)} + 1 \otimes X_{\sigma(1)} \otimes \delta^a_{\sigma(2)} \otimes Y_a \right) + \\
&\gamma_2 \left( 1 \otimes \delta^a_{\sigma(1)} \otimes X_{\sigma(2)} \otimes Y_a + 1 \otimes \delta^a_{\sigma(1)} \otimes Y_a \otimes X_{\sigma(2)} \right. \\
&\quad \left. + 1 \otimes \delta^a_{\sigma(1)} \otimes \delta^b_{\sigma(2)} Y_a \otimes Y_b + 1 \otimes \delta^a_{\sigma(1)} \otimes \delta^b_{\sigma(2)} \otimes Y_b Y_a \right) + \\
&\gamma_3 \left( -1 \otimes \delta^a_{\sigma(1)} \delta^b_{\sigma(2)} \otimes Y_b \otimes Y_a - 1 \otimes Y_b \otimes \delta^a_{\sigma(1)} \delta^b_{\sigma(2)} \otimes Y_a \right.
\end{aligned}$$

$$\begin{aligned}
& -1 \otimes \delta_{\sigma(1)}^a Y_b \otimes \delta_{\sigma(2)}^b \otimes Y_a - 1 \otimes \delta_{\sigma(1)}^a \otimes \delta_{\sigma(2)}^b Y_b \otimes Y_a \\
& - 1 \otimes \delta_{\sigma(2)}^b Y_b \otimes \delta_{\sigma(1)}^a \otimes Y_a - 1 \otimes \delta_{\sigma(2)}^b \otimes \delta_{\sigma(1)}^a Y_b \otimes Y_a \Big) + \\
& - \gamma_4 1 \otimes \Delta(\delta_{\sigma(1)\sigma(2)}^a) \otimes Y_a - r 1 \otimes \delta_{\sigma(1)}^a \otimes X_{\sigma(2)} \otimes Y_a \\
& + -r 1 \otimes X_{\sigma(1)} \otimes \delta_{\sigma(2)}^a \otimes Y_a + s 1 \otimes \delta_{\sigma(1)}^a \delta_{\sigma(2)}^b \otimes Y_b \otimes Y_a \\
& + s 1 \otimes Y_b \otimes \delta_{\sigma(1)}^a \delta_{\sigma(2)}^b \otimes Y_a + s 1 \otimes \delta_{\sigma(1)}^a Y_b \otimes \delta_{\sigma(2)}^b \otimes Y_a \\
& - s 1 \otimes \delta_{\sigma(1)}^b Y_b \otimes \delta_{\sigma(2)}^a \otimes Y_a + s 1 \otimes \delta_{\sigma(1)}^a \otimes \delta_{\sigma(2)}^b Y_b \otimes Y_a \\
& + (-r-s) 1 \otimes \delta_{\sigma(1)}^b \otimes \delta_{\sigma(2)}^a Y_b \otimes Y_a + \left(\frac{r}{2} + s\right) 1 \otimes \Delta(\delta_{\sigma(1)\sigma(2)}^a) \otimes Y_a \\
& - r 1 \otimes \delta_{\sigma(1)}^a \otimes \delta_{\sigma(2)}^b \otimes Y_b Y_a - \frac{r}{2} 1 \otimes \Delta(\delta_{\sigma(1)\sigma(2)}^a) \otimes Y_a = 0.
\end{aligned} \tag{3.56}$$

Hence we obtain (3.55).  $\square$

**Proposition 3.14.** *The cochain  $\psi : V \otimes \mathcal{A}^{\otimes 3} \rightarrow \mathbb{C}$  is a cyclic 2-cocycle if and only if  $\{\alpha_i \mid 1 \leq i \leq 2\}$ , satisfy (3.48),  $\{\beta_j \mid 1 \leq j \leq 9\}$ , fulfill (3.52), and  $\{\gamma_k \mid 1 \leq k \leq 4\}$  satisfy (3.55). The resulting cocycle is then a SAYD-twisted cyclic cocycle.*

*Proof.* We note that  $\psi$  is a Hochschild cocycle, i.e.  $b\psi = (b\psi)_0 + (b\psi)_1 + (b\psi)_2 = 0$ , if and only if  $(b\psi)_t = 0$ ,  $t = 0, 1, 2$ . We see that  $(b\psi)_2 = 0$  via Lemma 3.9,  $(b\psi)_1 = 0$  via Lemma 3.11,  $(b\psi)_0 = 0$  via Lemma 3.13.

On the other hand  $\psi$  is cyclic, i.e.  $\tau\psi = \psi$ , if and only if  $(\tau\psi)_t = \psi_t$ ,  $t = 0, 1, 2$ . Indeed, for  $t = 1$  Lemma 3.12, for  $t = 2$  Lemma 3.10 yields the claims. As for  $t = 0$  we have

$$\begin{aligned}
t\psi(1 \otimes a_0 \otimes a_1 \otimes a_2) &= \psi_0(1 \otimes a_2 \otimes a_0 \otimes a_1) - \psi_1(R^a \otimes Y_a(a_2) \otimes a_0 \otimes a_1) \\
&+ \frac{1}{2!} \psi_2(R^{ab} \otimes Y_b Y_a(a_2) \otimes a_0 \otimes a_1).
\end{aligned}$$

Accordingly,

$$t\psi(1 \otimes a_0 \otimes a_1 \otimes a_2) = (3.57) + (3.58) + (3.59) + (3.60).$$

$$\begin{aligned}
& r\tau\left(X_{\sigma(1)}(a_0)X_{\sigma(2)}(a_1)a_2\right) + r\tau\left(\delta_{\sigma(1)}^a(a_0)X_{\sigma(2)}Y_a(a_1)a_2\right) \\
& + s\tau\left(\delta_{\sigma(1)}^a\delta_{\sigma(2)}^bY_b(a_0)Y_a(a_1)a_2\right) + s\tau\left(\delta_{\sigma(1)\sigma(2)}^a(a_0)Y_a(a_1)a_2\right)
\end{aligned} \tag{3.57}$$

$$\begin{aligned}
& r\tau\left(\delta^a_{\sigma(1)}(a_0)X_{\sigma(2)}(a_1)Y_a(a_2)\right) + r\tau\left(X_{\sigma(1)}(a_0)\delta^a_{\sigma(2)}(a_1)Y_a(a_2)\right) \\
& - s\tau\left(\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}(a_0)Y_b(a_1)Y_a(a_2)\right) - s\tau\left(Y_b(a_0)\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}(a_1)Y_a(a_2)\right) \\
& + (r+s)\tau\left(\delta^b_{\sigma(1)}(a_0)\delta^a_{\sigma(2)}Y_b(a_1)Y_a(a_2)\right) - s\tau\left(\delta^a_{\sigma(1)}Y_b(a_0)\delta^b_{\sigma(2)}(a_1)Y_a(a_2)\right) \\
& + s\tau\left(\delta^b_{\sigma(1)}Y_b(a_0)\delta^a_{\sigma(2)}(a_1)Y_a(a_2)\right) - s\tau\left(\delta^a_{\sigma(1)}(a_0)\delta^b_{\sigma(2)}Y_b(a_1)Y_a(a_2)\right)
\end{aligned} \tag{3.58}$$

$$\left(-\frac{r}{2} - s\right)\tau\left(\Delta\left(\delta^a_{\sigma(1)\sigma(2)}\right)(a_0 \otimes a_1)Y_a(a_2)\right) \tag{3.59}$$

$$\frac{r}{2!}\tau\left(\delta^a_{\sigma(1)}(a_0)\delta^b_{\sigma(2)}(a_1)Y_bY_a(a_2)\right) + \frac{r}{2!}\tau\left(\delta^b_{\sigma(1)}(a_0)\delta^a_{\sigma(2)}(a_1)Y_bY_a(a_2)\right) \tag{3.60}$$

We shall put the above expressions into the standard form  $\tau(a_0 h^1(a_1) h^2(a_2))$ . To this end we use the integration by parts property (3.37). On the computation below, we will employ the actions

$$(f \otimes g) \triangleleft Y := f \otimes g \cdot Y, \quad (f \otimes g) \blacktriangleleft Y := f \cdot Y \otimes g. \tag{3.61}$$

of  $\mathfrak{g}_0$  on  $\mathcal{H}_n^{\otimes 2}$ . We first deal with (3.59) by

$$\begin{aligned}
& \tau\left(\Delta\left(\delta^a_{\sigma(1)\sigma(2)}\right)(a_0 \otimes a_1)Y_a(a_2)\right) = \\
& - 2\tau\left(a_0\delta^a_{\sigma(1)\sigma(2)}(a_1)Y_a(a_2)\right) - \tau\left(a_0\left(\Delta\left(\delta^a_{\sigma(1)\sigma(2)}\right)\triangleleft Y_a\right)(a_1 \otimes a_2)\right).
\end{aligned} \tag{3.62}$$

As for (3.60) we compute

$$\begin{aligned}
& \frac{1}{2!}\tau\left(\delta^a_{\sigma(1)}(a_0)\delta^b_{\sigma(2)}(a_1)Y_bY_a(a_2)\right) + \frac{1}{2!}\tau\left(\delta^b_{\sigma(1)}(a_0)\delta^a_{\sigma(2)}(a_1)Y_bY_a(a_2)\right) = \\
& - \frac{1}{2}\tau\left(a_0\delta^b_{\sigma(2)}(a_1)\delta^a_{\sigma(1)}Y_bY_a(a_2)\right) - \frac{1}{2}\tau\left(a_0\delta^a_{\sigma(2)}(a_1)\delta^b_{\sigma(1)}Y_bY_a(a_2)\right) = \\
& \frac{1}{2}\tau\left(a_0\delta^a_{\sigma(1)}(a_1)\delta^b_{\sigma(2)}Y_bY_a(a_2)\right) + \frac{1}{2}\tau\left(a_0\delta^a_{\sigma(1)}(a_1)\delta^b_{\sigma(2)}Y_aY_b(a_2)\right) = \\
& \tau\left(a_0\delta^a_{\sigma(1)}(a_1)\delta^b_{\sigma(2)}Y_bY_a(a_2)\right) + \frac{1}{2}\tau\left(a_0\delta^a_{\sigma(1)}(a_1)\delta^b_{\sigma(2)}[Y_a, Y_b](a_2)\right) = \\
& \tau\left(a_0\delta^a_{\sigma(1)}(a_1)\delta^b_{\sigma(2)}Y_bY_a(a_2)\right) + \frac{1}{2}\tau\left(a_0\left(\Delta\left(\delta^a_{\sigma(1)\sigma(2)}\right)\triangleleft Y_a\right)(a_1 \otimes a_2)\right).
\end{aligned} \tag{3.63}$$

On the third step we compute (3.57) term by term.

$$\begin{aligned} \tau\left(X_{\sigma(1)}(a_0)X_{\sigma(2)}(a_1)a_2\right) = \\ -\tau\left(a_0X_{\sigma(2)}(a_1)X_{\sigma(1)}(a_2)\right) + \tau\left(a_0\delta^a_{\sigma(1)}Y_aX_{\sigma(2)}(a_1)a_2\right) \\ + \tau\left(a_0X_{\sigma(2)}(a_1)\delta^a_{\sigma(1)}Y_a(a_2)\right) + \tau\left(a_0Y_aX_{\sigma(2)}(a_1)\delta^a_{\sigma(1)}(a_2)\right). \end{aligned}$$

$$\begin{aligned} \tau\left(\delta^a_{\sigma(1)}(a_0)X_{\sigma(2)}Y_a(a_1)a_2\right) = \\ -\tau\left(a_0\delta^a_{\sigma(1)}X_{\sigma(2)}Y_a(a_1)a_2\right) - \tau\left(a_0X_{\sigma(2)}Y_a(a_1)\delta^a_{\sigma(1)}(a_2)\right). \end{aligned}$$

$$\begin{aligned} \tau\left(\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}Y_b(a_0)Y_a(a_1)a_2\right) = \\ \tau\left(Y_b(a_0)\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}Y_a(a_1)a_2\right) + \tau\left(Y_b(a_0)Y_a(a_1)\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}(a_2)\right) \\ + \tau\left(Y_b(a_0)\delta^a_{\sigma(1)}Y_a(a_1)\delta^b_{\sigma(2)}(a_2)\right) + \tau\left(Y_b(a_0)\delta^b_{\sigma(2)}Y_a(a_1)\delta^a_{\sigma(1)}(a_2)\right) = \\ -\tau\left(a_0\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}Y_bY_a(a_1)a_2\right) - 2\tau\left(a_0\delta^a_{\sigma(1)\sigma(2)}Y_a(a_1)a_2\right) \\ -\tau\left(a_0\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}Y_a(a_1)Y_b(a_2)\right) - \tau\left(a_0Y_bY_a(a_1)\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}(a_2)\right) \\ -\tau\left(a_0Y_a(a_1)\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}Y_b(a_2)\right) - 2\tau\left(a_0Y_a(a_1)\delta^a_{\sigma(1)\sigma(2)}(a_2)\right) \\ -\tau\left(a_0\delta^a_{\sigma(1)}Y_bY_a(a_1)\delta^b_{\sigma(2)}(a_2)\right) - \tau\left(a_0\left(\Delta\left(\delta^a_{\sigma(1)\sigma(2)}\right)\blacktriangleleft Y_a\right)(a_1\otimes a_2)\right) \\ -\tau\left(a_0\delta^a_{\sigma(1)}Y_a(a_1)\delta^b_{\sigma(2)}Y_b(a_2)\right) - \tau\left(a_0\delta^b_{\sigma(2)}Y_bY_a(a_1)\delta^a_{\sigma(1)}(a_2)\right) \\ -\tau\left(a_0\delta^b_{\sigma(2)}Y_a(a_1)\delta^a_{\sigma(1)}Y_b(a_2)\right) - \tau\left(a_0\left(\Delta\left(\delta^a_{\sigma(1)\sigma(2)}\right)\blacktriangleleft Y_a\right)(a_1\otimes a_2)\right). \end{aligned}$$

$$\begin{aligned} \tau\left(\delta^a_{\sigma(1)\sigma(2)}(a_0)Y_a(a_1)a_2\right) = \\ \tau\left(a_0\delta^a_{\sigma(1)\sigma(2)}Y_a(a_1)a_2\right) + \tau\left(a_0Y_a(a_1)\delta^a_{\sigma(1)\sigma(2)}(a_2)\right) \\ + \tau\left(a_0\left(\Delta\left(\delta^a_{\sigma(1)\sigma(2)}\right)\blacktriangleleft Y_a\right)(a_1\otimes a_2)\right). \end{aligned}$$

Summing up, we obtain

$$\begin{aligned}
& r\tau\left(X_{\sigma(1)}(a_0)X_{\sigma(2)}(a_1)a_2\right) + r\tau\left(\delta^a_{\sigma(1)}(a_0)X_{\sigma(2)}Y_a(a_1)a_2\right) \\
& + s\tau\left(\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}Y_b(a_0)Y_a(a_1)a_2\right) + s\tau\left(\delta^a_{\sigma(1)\sigma(2)}(a_0)Y_a(a_1)a_2\right) = \\
& - r\tau\left(a_0X_{\sigma(2)}(a_1)X_{\sigma(1)}(a_2)\right) - s\tau\left(a_0\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}Y_a(a_1)Y_b(a_2)\right) \\
& + r\tau\left(a_0X_{\sigma(2)}(a_1)\delta^a_{\sigma(1)}Y_a(a_2)\right) - s\tau\left(a_0Y_a(a_1)\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}Y_b(a_2)\right) \\
& - s\tau\left(a_0\delta^a_{\sigma(1)}Y_a(a_1)\delta^b_{\sigma(2)}Y_b(a_2)\right) - s\tau\left(a_0\delta^b_{\sigma(2)}Y_a(a_1)\delta^a_{\sigma(1)}Y_b(a_2)\right).
\end{aligned} \tag{3.64}$$

Finally, for (3.58) we have

$$\begin{aligned}
& \tau\left(\delta^a_{\sigma(1)}(a_0)X_{\sigma(2)}(a_1)Y_a(a_2)\right) = \\
& - \tau\left(a_0\delta^a_{\sigma(1)}X_{\sigma(2)}(a_1)Y_a(a_2)\right) - \tau\left(a_0X_{\sigma(2)}(a_1)\delta^a_{\sigma(1)}Y_a(a_2)\right),
\end{aligned}$$

$$\begin{aligned}
& \tau\left(X_{\sigma(1)}(a_0)\delta^a_{\sigma(2)}(a_1)Y_a(a_2)\right) = \\
& - \tau\left(a_0X_{\sigma(1)}\delta^a_{\sigma(2)}(a_1)Y_a(a_2)\right) - \tau\left(a_0\delta^a_{\sigma(2)}(a_1)X_{\sigma(1)}Y_a(a_2)\right) \\
& + \tau\left(a_0\delta^b_{\sigma(1)}\delta^a_{\sigma(2)}Y_b(a_1)Y_a(a_2)\right) - 2\tau\left(a_0\delta^a_{\sigma(1)\sigma(2)}(a_1)Y_a(a_2)\right) \\
& + \tau\left(a_0\delta^a_{\sigma(2)}(a_1)\delta^b_{\sigma(1)}Y_bY_a(a_2)\right) + \tau\left(a_0\delta^a_{\sigma(2)}Y_b(a_1)\delta^b_{\sigma(1)}Y_a(a_2)\right) \\
& - \tau\left(a_0\left(\Delta\left(\delta^a_{\sigma(1)\sigma(2)}\right)\triangleleft Y_a\right)(a_1\otimes a_2)\right).
\end{aligned}$$

$$\begin{aligned}
& \tau\left(\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}(a_0)Y_b(a_1)Y_a(a_2)\right) = \\
& \tau\left(a_0\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}Y_b(a_1)Y_a(a_2)\right) + \tau\left(a_0Y_b(a_1)\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}Y_a(a_2)\right) \\
& \tau\left(a_0\delta^a_{\sigma(1)}Y_b(a_1)\delta^b_{\sigma(2)}Y_a(a_2)\right) + \tau\left(a_0\delta^b_{\sigma(2)}Y_b(a_1)\delta^a_{\sigma(1)}Y_a(a_2)\right).
\end{aligned}$$

$$\begin{aligned}
& \tau\left(\delta^b_{\sigma(1)}(a_0)\delta^a_{\sigma(2)}Y_b(a_1)Y_a(a_2)\right) = \\
& \tau\left(a_0\delta^b_{\sigma(1)}\delta^a_{\sigma(2)}Y_b(a_1)Y_a(a_2)\right) + \tau\left(a_0\delta^a_{\sigma(2)}Y_b(a_1)\delta^b_{\sigma(1)}Y_a(a_2)\right).
\end{aligned}$$

$$\begin{aligned}
& \tau\left(\delta^a_{\sigma(1)}Y_b(a_0)\delta^b_{\sigma(2)}(a_1)Y_a(a_2)\right) = \\
& -\tau\left(Y_b(a_0)\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}(a_1)Y_a(a_2)\right) + \tau\left(a_0\delta^b_{\sigma(2)}Y_b(a_1)\delta^a_{\sigma(1)}Y_a(a_2)\right) \\
& + \tau\left(a_0\delta^b_{\sigma(2)}(a_1)\delta^a_{\sigma(1)}Y_bY_a(a_2)\right) + \tau\left(a_0\left(\Delta\left(\delta^a_{\sigma(1)\sigma(2)}\right)\triangleleft Y_a\right)(a_1\otimes a_2)\right).
\end{aligned}$$

$$\begin{aligned}
& \tau\left(\delta^b_{\sigma(1)}Y_b(a_0)\delta^a_{\sigma(2)}(a_1)Y_a(a_2)\right) = \\
& \tau\left(a_0\delta^b_{\sigma(1)}\delta^a_{\sigma(2)}Y_b(a_1)Y_a(a_2)\right) - 2\tau\left(a_0\delta^a_{\sigma(1)\sigma(2)}(a_1)Y_a(a_2)\right) \\
& + \tau\left(a_0\delta^b_{\sigma(1)}\delta^a_{\sigma(2)}(a_1)Y_bY_a(a_2)\right) + \tau\left(a_0\delta^a_{\sigma(2)}Y_b(a_1)\delta^b_{\sigma(1)}Y_a(a_2)\right) \\
& - \tau\left(a_0\left(\Delta\left(\delta^a_{\sigma(1)\sigma(2)}\right)\triangleleft Y_a\right)(a_1\otimes a_2)\right) + \tau\left(a_0\delta^a_{\sigma(2)}(a_1)\delta^b_{\sigma(1)}Y_bY_a(a_2)\right).
\end{aligned}$$

$$\begin{aligned}
& \tau\left(\delta^a_{\sigma(1)}(a_0)\delta^b_{\sigma(2)}Y_b(a_1)Y_a(a_2)\right) = \\
& -\tau\left(a_0\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}Y_b(a_1)Y_a(a_2)\right) - \tau\left(a_0\delta^b_{\sigma(2)}Y_b(a_1)\delta^a_{\sigma(1)}Y_a(a_2)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& r\tau\left(\delta^a_{\sigma(1)}(a_0)X_{\sigma(2)}(a_1)Y_a(a_2)\right) + r\tau\left(X_{\sigma(1)}(a_0)\delta^a_{\sigma(2)}(a_1)Y_a(a_2)\right) \\
& - s\tau\left(\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}(a_0)Y_b(a_1)Y_a(a_2)\right) - s\tau\left(Y_b(a_0)\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}(a_1)Y_a(a_2)\right) \\
& + (r+s)\tau\left(\delta^b_{\sigma(1)}(a_0)\delta^a_{\sigma(2)}Y_b(a_1)Y_a(a_2)\right) - s\tau\left(\delta^a_{\sigma(1)}Y_b(a_0)\delta^b_{\sigma(2)}(a_1)Y_a(a_2)\right) \\
& + s\tau\left(\delta^b_{\sigma(1)}Y_b(a_0)\delta^a_{\sigma(2)}(a_1)Y_a(a_2)\right) - s\tau\left(\delta^a_{\sigma(1)}(a_0)\delta^b_{\sigma(2)}Y_b(a_1)Y_a(a_2)\right) \\
& = -r\tau\left(a_0\delta^a_{\sigma(2)}(a_1)X_{\sigma(1)}Y_a(a_2)\right) - r\tau\left(a_0X_{\sigma(2)}(a_1)\delta^a_{\sigma(1)}Y_a(a_2)\right) \\
& - s\tau\left(a_0Y_b(a_1)\delta^a_{\sigma(1)}\delta^b_{\sigma(2)}Y_a(a_2)\right) - s\tau\left(a_0\delta^b_{\sigma(2)}Y_b(a_1)\delta^a_{\sigma(1)}Y_a(a_2)\right) \\
& + s\tau\left(a_0\delta^a_{\sigma(2)}Y_b(a_1)\delta^b_{\sigma(1)}Y_a(a_2)\right) - r\tau\left(a_0\delta^a_{\sigma(1)}(a_1)\delta^b_{\sigma(2)}Y_bY_a(a_2)\right) \\
& (-r-s)\tau\left(a_0\left(\Delta\left(\delta^a_{\sigma(1)\sigma(2)}\right)\triangleleft Y_a\right)(a_1\otimes a_2)\right) + (-r-s)\tau\left(a_0\delta^a_{\sigma(1)\sigma(2)}(a_1)Y_a(a_2)\right).
\end{aligned} \tag{3.65}$$

As a result of the computations (3.62), (3.63), (3.64), and (3.65) we obtain

$$t\psi(1\otimes a_0\otimes a_1\otimes a_2) = \psi_0(1\otimes a_0\otimes a_1\otimes a_2).$$

□

**Theorem 3.15.** *The following cochain  $\varphi = \varphi_0 + \varphi_1 + \varphi_2 \in C_{\mathcal{K}}^2(\mathcal{A}, V)$  is a SAYD-twisted cyclic cocycle and cohomologous to  $\psi$  which is defined in Proposition 3.14.*

$$\varphi_2 = \chi_\tau^{\text{eq}} \left( S_{ab} \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^a \otimes \delta_{\sigma(2)}^b + S_{ab} \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^b \otimes \delta_{\sigma(2)}^a \right) \quad (3.66)$$

$$\begin{aligned} \varphi_1 = \chi_\tau^{\text{eq}} \Big( & -S_a \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^a \otimes X_{\sigma(2)} - S_a \otimes \mathbf{1} \otimes X_{\sigma(1)} \otimes \delta_{\sigma(2)}^a \\ & - S_a \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^b \otimes \delta_{\sigma(2)}^a Y_b + \frac{1}{2} S_a \otimes \mathbf{1} \otimes \Delta(\delta_{\sigma(1)\sigma(2)}^a) \Big) \end{aligned} \quad (3.67)$$

$$\varphi_0 = \chi_\tau^{\text{eq}} \left( 1^* \otimes \mathbf{1} \otimes X_{\sigma(1)} \otimes X_{\sigma(2)} + 1^* \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^a \otimes X_{\sigma(2)} Y_a \right), \quad (3.68)$$

*Proof.* As a result of Proposition 3.14 we can write  $\psi = r\varphi + s\phi$  for a 2-cochain  $\phi = \phi_0 + \phi_1 + \phi_2$  given by

$$\phi_2 = 0 \quad (3.69)$$

$$\begin{aligned} \phi_1 = \chi_\tau^{\text{eq}} \Big( & S_a \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^a \delta_{\sigma(2)}^b \otimes Y_b + S_a \otimes \mathbf{1} \otimes Y_b \otimes \delta_{\sigma(1)}^a \delta_{\sigma(2)}^b \\ & + S_a \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^a Y_b \otimes \delta_{\sigma(2)}^b - S_a \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^b Y_b \otimes \delta_{\sigma(2)}^a \\ & + S_a \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^a \otimes \delta_{\sigma(2)}^b Y_b - S_a \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^b \otimes \delta_{\sigma(2)}^a Y_b \\ & + S_a \otimes \mathbf{1} \otimes \Delta(\delta_{\sigma(1)\sigma(2)}^a) \Big) \end{aligned} \quad (3.70)$$

$$\phi_0 = \chi_\tau^{\text{eq}} \left( 1^* \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^a \delta_{\sigma(2)}^b Y_b \otimes Y_a + 1^* \otimes \mathbf{1} \otimes \delta_{\sigma(1)\sigma(2)}^a \otimes Y_a \right). \quad (3.71)$$

We note that

$$\phi_1 = \chi_\tau^{\text{eq}} \left( S_a \otimes \mathbf{1} \otimes \Delta \left( \delta_{\sigma(1)}^a \delta_{\sigma(2)}^b Y_b \right) + S_a \otimes \mathbf{1} \otimes \Delta(\delta_{\sigma(1)\sigma(2)}^a) \right). \quad (3.72)$$

It is then straightforward to check that the 1-cochain  $\phi' = \phi'_0 + \phi'_1 + \phi'_2$  given by

$$\phi'_2 = 0 \quad (3.73)$$

$$\phi'_1 = \chi_\tau^{\text{eq}} \left( S_a \otimes \mathbf{1} \otimes \delta_{\sigma(1)}^a \delta_{\sigma(2)}^b Y_b + S_a \otimes \mathbf{1} \otimes \delta_{\sigma(1)\sigma(2)}^a \right) \quad (3.74)$$

$$\phi'_0 = 0 \quad (3.75)$$

is an equivariant cyclic 1-cocycle, and that

$$b\phi' = \phi. \quad (3.76)$$

□



## 4 The characteristic map with coefficients

In this section we construct a new characteristic map from the truncated Weil complex of the Lie algebra  $\mathfrak{g}_0 := g\ell_n$  to the cyclic complex of the crossed product algebra  $\mathcal{A} = C_c^\infty(F^+) \rtimes \Gamma$ , and we illustrate it completely in codimensions  $n = 1$  and  $n = 2$ . We observe that the resulting cocycles in codimension 1 match with those in [2, 3] by Connes-Moscovici .

Such a characteristic map is obtained by composing a series of maps

$$H(W(\mathfrak{g}_0, V)) \xrightarrow{\mathfrak{D}_P} H(C(\mathfrak{g}_0, V)) \xrightarrow{\cong} HC(U(\mathfrak{g}_0), V) \xrightarrow{\chi_\varphi} HC(\mathcal{A}). \quad (4.1)$$

As it is shown in [13] the truncated Weil algebra is identical with  $W(\mathfrak{g}_0, V)$ . The Poincaré isomorphism  $\mathfrak{D}_P$  is defined in [13, Prop. 4.4]. The middle quasi-isomorphism is defined in [13, Thm. 6.2]. Finally the map  $\chi_\varphi$  is given by the cup product, in the sense of [8, 12], with the SAYD-twisted cyclic cocycle  $\varphi$  defined in Theorem 3.15.

Let us recall the mentioned cup product from [12]. Let  $C$  be a  $H$ -module coalgebra and  $A$  be an  $H$ -module algebra that are equipped with a mapping

$$C \otimes A \rightarrow A, \quad c \otimes a \mapsto c(a) \quad (4.2)$$

satisfying the conditions

$$(h \cdot c)(a) = h \cdot (c(a)), \quad c(ab) = c_{(1)}(a)c_{(2)}(b), \quad c(1) = \varepsilon(c)1. \quad (4.3)$$

Let also  $V$  be a SAYD module over a Hopf algebra  $H$ . One defines

$$\cup : C_H^p(C, V) \otimes C_H^q(A, V) \rightarrow C^{p+q}(A) \quad (4.4)$$

for any  $\varphi \in C_H^q(A, V)$  and any  $x = v \otimes_H c^0 \otimes \cdots \otimes c^p \in C_H^p(C, V)$ ,

$$(x \cup \varphi)(a_0 \otimes \cdots \otimes a_{p+q}) := \sum_{\sigma \in \text{Sh}(p, q)} (-1)^\sigma \partial_{\bar{\sigma}(p)} \cdots \partial_{\bar{\sigma}(1)} \varphi(\langle \partial_{\bar{\sigma}(p+q)} \cdots \partial_{\bar{\sigma}(p+1)} x, a_0 \otimes \cdots \otimes a_{p+q} \rangle), \quad (4.5)$$

where

$$\langle x, a_0 \otimes \cdots \otimes a_n \rangle := v \otimes_H c^0(a^0) \otimes \cdots \otimes c^n(a^n). \quad (4.6)$$

Here  $\text{Sh}(p, q)$  is the set of all  $(p, q)$ -shuffle permutations, and

$$\bar{\sigma}(n) = \sigma(n) - 1. \quad (4.7)$$

We set

$$C := H := \mathcal{K}, \quad V = S(\mathfrak{g}_0^*_{[2n]}), \quad A = \mathcal{A}.$$

Here  $H$  acts on  $C$  via multiplication, on  $A$  as (3.22), and on  $V$  via the coadjoint action. This construction yields for any  $\varphi \in C_{\mathcal{K}}^n(\mathcal{A}, V)$  a characteristic map

$$\chi_{\varphi} : C_{\mathcal{K}}^{\ell}(\mathcal{K}, V) \rightarrow C^{\ell+n}(\mathcal{A}). \quad (4.8)$$

## 4.1 The characteristic map in codimension 1

In this subsection we apply the SAYD-twisted cyclic cocycle (3.35) to illustrate the new characteristic map

$$\chi_{\varphi} : H(W(g\ell_1)_{[2]}) \longrightarrow HC(C_c^{\infty}(F^+\mathbb{R}) \rtimes \Gamma). \quad (4.9)$$

In order to verify that the new characteristic map is geometrically meaningful, we compare it with the Connes-Moscovici computations for the classes of  $\mathcal{H}_1$  in [3]. For the convenience of the reader we recall the Hopf-cyclic classes in  $C(\mathcal{H}_1, \mathbb{C}_{\delta})$ , namely the the transverse fundamental class

$$\text{TF} = X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y \in C^2(\mathcal{H}_1, \mathbb{C}_{\delta}) \quad (4.10)$$

and the Godbillon-Vey class

$$\text{GV} = \delta_1 \in C^1(\mathcal{H}_1, \mathbb{C}_{\delta}). \quad (4.11)$$

In view of the characteristic map (2.14), one expresses the characteristic classes  $TF \in C^2(\mathcal{A}_{\Gamma})$  and  $GV \in C^1(\mathcal{A}_{\Gamma})$  as

$$\begin{aligned} TF(a_0 \otimes a_1 \otimes a_2) \\ = \tau(a_0 X(a_1) Y(a_2)) - \tau(a_0 Y(a_1) X(a_2)) - \tau(a_0 \delta_1 Y(a_1) Y(a_2)) \end{aligned} \quad (4.12)$$

and

$$GV(a_0 \otimes a_1) = \tau(a_0 \delta_1(a_1)). \quad (4.13)$$

In this subsection we set  $\mathcal{K} = U(g\ell_1)$ ,  $V = S(g\ell_1^*_{[2]})$ , and  $\mathcal{A} = C_c^{\infty}(F^+\mathbb{R}) \rtimes \Gamma$ .

The next step is to find the representative cocycles of  $H(g\ell_1, V)$ . Let  $\{Y\}$  and  $\{\theta\}$  be a dual pair of bases for  $g\ell_1$  and  $g\ell_1^*$ .

By the Vey basis [4], the cohomology of  $W(g\ell_1)_{[2]}$  is spanned by

$$\text{TF} := 1 \in S(g\ell_1^*)_{[2]}, \quad \text{GV} := \theta \otimes R \in g\ell_1^* \otimes S(g\ell_1^*)_{[2]} \quad (4.14)$$

Applying the Poincaré duality [13, Prop. 4.4], we obtain the classes in  $HC(g\ell_1, V)$

$$\mathfrak{D}_P(1) = Y \otimes 1 \in g\ell_1 \otimes V, \quad \mathfrak{D}_P(\theta \otimes R) = R \in V. \quad (4.15)$$

**Proposition 4.1.** *The Hopf-cyclic cohomology  $HC(\mathcal{K}, V)$  is generated by the classes*

$$[R] \in HC^0(\mathcal{K}, V), \quad (4.16)$$

$$[1 \otimes Y + \frac{1}{2}R \otimes Y^2] \in HC^1(\mathcal{K}, V). \quad (4.17)$$

*Proof.* We first check that  $R \in C^0(\mathcal{K}, V)$  is a cyclic 0-cocycle. Indeed,

$$b(R) = R \otimes 1 - R \otimes 1 = 0, \quad \mathfrak{t}(R) = R. \quad (4.18)$$

In a similar fashion,

$$b(1 \otimes Y) = 1 \otimes 1 \otimes Y - 1 \otimes \Delta(Y) + 1 \otimes Y \otimes 1 + R \otimes Y \otimes Y$$

yields

$$b(1 \otimes Y + \frac{1}{2}R \otimes Y^2) = 0. \quad (4.19)$$

Moreover, applying the cyclic map we observe

$$\mathfrak{t}(1 \otimes Y + \frac{1}{2}R \otimes Y^2) = -1 \otimes Y - R \otimes Y^2 + \frac{1}{2}R \otimes Y^2 = -(1 \otimes Y + \frac{1}{2}R \otimes Y^2). \quad (4.20)$$

Finally, we apply the quasi-isomorphism

$$\mu : C^\bullet(U(\mathfrak{g}), V) \longrightarrow C_\bullet(\mathfrak{g}, V), \quad (4.21)$$

which, for any Lie algebra  $\mathfrak{g}$ , is the left inverse of the anti-symmetrization map, see [3, 13]. Then we have

$$\mu(R) = R, \quad \mu(1 \otimes Y + \frac{1}{2}R \otimes Y^2) = 1 \otimes Y, \quad (4.22)$$

the generators of the cohomology  $HC(g\ell_1, V)$ . This observation finishes the proof.  $\square$

Let us find the images of the cocycles (4.16) and (4.17) under the map

$$\chi_\varphi : C_{\mathcal{K}}^\bullet(\mathcal{K}, V) \rightarrow C^{\bullet+1}(\mathcal{A}). \quad (4.23)$$

We compute

$$\begin{aligned} \chi_\varphi(\theta)(a_0 \otimes a_1) &= (\varphi \cup (\theta \otimes 1))(a_0 \otimes a_1) \\ &= \varphi(\langle \partial_0(\theta \otimes 1), a_0 \otimes a_1 \rangle) = \varphi(\theta \otimes a_0 \otimes a_1) = -\tau(a_0 \delta_1(a_1)), \end{aligned} \quad (4.24)$$

and in the same way,

$$\begin{aligned} \chi_\varphi(1 \otimes Y + \frac{1}{2}\theta \otimes Y^2)(a_0 \otimes a_1 \otimes a_2) &= \\ \sum_{\sigma \in Sh(1,1)} (-1)^\sigma \partial_{\bar{\sigma}(1)} \varphi(\langle \partial_{\bar{\sigma}(2)}(1 \otimes Y + \frac{1}{2}\theta \otimes Y^2), a_0 \otimes a_1 \otimes a_2 \rangle) &= \\ -\partial_0 \varphi(\langle \partial_1(1 \otimes Y + \frac{1}{2}\theta \otimes Y^2), a_0 \otimes a_1 \otimes a_2 \rangle) + \\ \partial_1 \varphi(\langle \partial_0(1 \otimes Y + \frac{1}{2}\theta \otimes Y^2), a_0 \otimes a_1 \otimes a_2 \rangle). \end{aligned} \quad (4.25)$$

Then since

$$\begin{aligned} \partial_1(1 \otimes Y + \frac{1}{2}\theta \otimes Y^2) &= 1 \otimes 1 \otimes 1 \otimes Y + 1 \otimes 1 \otimes Y \otimes 1 + \\ \frac{1}{2}\theta \otimes 1 \otimes Y^2 \otimes 1 + \frac{1}{2}\theta \otimes 1 \otimes 1 \otimes Y^2 + \theta \otimes 1 \otimes Y \otimes Y, \end{aligned} \quad (4.26)$$

we obtain

$$\begin{aligned} \chi_\varphi(1 \otimes Y + \frac{1}{2}\theta \otimes Y^2)(a_0 \otimes a_1 \otimes a_2) &= \\ \varphi(-1 \otimes a_0 a_1 \otimes Y(a_2) - 1 \otimes a_0 Y(a_1) \otimes a_2 - \frac{1}{2}\theta \otimes a_0 Y^2(a_1) \otimes a_2 \\ - \frac{1}{2}\theta \otimes a_0 a_1 \otimes Y^2(a_2) - \theta \otimes a_0 Y(a_1) \otimes Y(a_2)) + \varphi(1 \otimes a_0 \otimes a_1 Y(a_2) \\ + \frac{1}{2}\theta \otimes a_0 \otimes a_1 Y^2(a_2)) \\ = -\tau(a_0 Y(a_1) X(a_2)) + \tau(a_0 X(a_1) Y(a_2)) + \frac{1}{2}\tau(a_0 Y^2(a_1) \delta_1(a_2)) \\ + \frac{1}{2}\tau(a_0 \delta_1(a_1) Y^2(a_2)) + \tau(a_0 Y(a_1) \delta_1 Y(a_2)). \end{aligned} \quad (4.27)$$

**Remark 4.2.** We note that

$$b(\delta_1 Y^2) \in C_{\mathcal{H}_1}^2(\mathcal{H}_1, \mathbb{C}_\delta) \quad (4.28)$$

is a cyclic cocycle and

$$\chi_\varphi(1 \otimes Y + \frac{1}{2}\theta \otimes Y^2) = \chi_\tau(\text{TF} + \frac{1}{2}b(\delta_1 Y^2)). \quad (4.29)$$

Hence, we obtain the transverse fundamental class up to a coboundary. Similarly we have

$$\chi_\varphi(\theta) = -\chi_\tau(\text{GV}), \quad (4.30)$$

*i.e.* we also obtain the Godbillon-Vey class.

## 4.2 The characteristic map in codimension 2

In this subsection we exercise the machinery we developed in Subsection 3.3 for  $n = 2$ . There is no such computations in the literature that we know of.

We keep our conventions as before, *i.e.*  $\mathcal{K} := U(g\ell_2)$ ,  $V := S(g\ell_2^*)_{[4]}$ ,  $\mathcal{H} := \mathcal{K}_2$ , and  $\mathcal{A} := C_c^\infty(F^+\mathbb{R}^2) \rtimes \Gamma$ .

Let us recall the Vey basis of the cohomology of the truncated Weil algebra  $W(g\ell_2)_{[4]}$ . To this end, we fix the following notation

$$\begin{aligned} c_1 &= \text{Tr} = R_1^1 + R_2^2 \in S(g\ell_2^*), & c_2 &= R_2^1 R_1^2 \in S(g\ell_2^*), \\ u_1 &= \theta_1^1 + \theta_2^2, & u_2 &= \theta_1^1 \wedge \theta_2^1 \wedge \theta_1^2, & \omega &= \theta_1^1 \wedge \theta_2^1 \wedge \theta_1^2 \wedge \theta_2^2. \end{aligned}$$

The Vey basis [4], is then introduced by

$$\{1, c_1^2 \otimes u_1, c_2 \otimes u_1, c_2 \otimes u_2, c_1^2 \otimes \omega, c_2 \otimes \omega\}. \quad (4.31)$$

Next, the Poincaré duality yields the 6 cocycles in the complex  $C(g\ell_2, V)$ :

$$\begin{aligned} \mathfrak{D}_P(1) &= 1 \otimes Y_1^1 \wedge Y_1^2 \wedge Y_2^1 \wedge Y_2^2, \\ \mathfrak{D}_P(c_2 \otimes u_1) &= c_2 \otimes (Y_1^2 \wedge Y_2^1 \wedge Y_2^2 - Y_1^1 \wedge Y_1^2 \wedge Y_2^1), \\ \mathfrak{D}_P(c_1^2 \otimes u_1) &= c_1^2 \otimes (Y_1^2 \wedge Y_2^1 \wedge Y_2^2 - Y_1^1 \wedge Y_1^2 \wedge Y_2^1), \\ \mathfrak{D}_P(c_2 \otimes u_2) &= c_2 \otimes Y_2^2, & \mathfrak{D}_P(c_1^2 \otimes \omega) &= c_1^2, \\ \mathfrak{D}_P(c_2 \otimes \omega) &= c_2. \end{aligned}$$

We label  $Y_i^j$  as

$$Y_1 := Y_1^1, \quad Y_2 := Y_1^2, \quad Y_3 := Y_2^1, \quad Y_4 := Y_2^2. \quad (4.32)$$

**Proposition 4.3.** *The Hopf-cyclic cohomology  $HC(\mathcal{K}, V)$  is generated by the classes*

$$\begin{aligned}
[\mathcal{T}\mathcal{F}] &\in HC^4(\mathcal{K}, V) \\
[\mathcal{G}\mathcal{V}] &:= \left[ \sum_{\sigma \in S_3} (-1)^\sigma c_1^2 \otimes \left( Y_{\sigma(2)} \otimes Y_{\sigma(3)} \otimes Y_{\sigma(4)} \right. \right. \\
&\quad \left. \left. - Y_{\sigma(1)} \otimes Y_{\sigma(2)} \otimes Y_{\sigma(3)} \right) \right] \in HC^3(\mathcal{K}, V), \\
[\mathcal{R}_1] &:= \left[ \sum_{\sigma \in S_3} (-1)^\sigma c_2 \otimes \left( Y_{\sigma(2)} \otimes Y_{\sigma(3)} \otimes Y_{\sigma(4)} \right. \right. \\
&\quad \left. \left. - Y_{\sigma(1)} \otimes Y_{\sigma(2)} \otimes Y_{\sigma(3)} \right) \right] \in HC^3(\mathcal{K}, V), \\
[\mathcal{R}_2] &:= [c_2 \otimes Y_4] \in HC^1(\mathcal{K}, V), \\
[\mathcal{R}_3] &:= [c_1^2] \in HC^0(\mathcal{K}, V), \\
[\mathcal{R}_4] &:= [c_2] \in HC^0(\mathcal{K}, V).
\end{aligned} \tag{4.33}$$

*Proof.* It is straightforward to check that  $\mathcal{R}_1, \dots, \mathcal{R}_4$  and  $\mathcal{G}\mathcal{V}$  are Hopf-cyclic cocycles and that

$$\begin{aligned}
\mu(\mathcal{R}_1) &= \mathfrak{D}_P(c_2 \otimes u_1), & \mu(\mathcal{R}_2) &= \mathfrak{D}_P(c_2 \otimes u_2), \\
\mu(\mathcal{R}_3) &= \mathfrak{D}_P(c_1^2 \otimes \omega), & \mu(\mathcal{R}_4) &= \mathfrak{D}_P(c_2 \otimes \omega), \\
\mu(\mathcal{G}\mathcal{V}) &= \mathfrak{D}_P(c_1^2 \otimes u_1).
\end{aligned}$$

On the other hand,

$$[\mathcal{T}\mathcal{F}] = \left[ \sum_{\sigma \in S_4} (-1)^\sigma 1 \otimes Y_{\sigma(1)} \otimes Y_{\sigma(2)} \otimes Y_{\sigma(3)} \otimes Y_{\sigma(4)} \right] \in E_1^{2,2}(U(gl_2), V) \tag{4.34}$$

is a cyclic cocycle in the  $E_1$  level of the spectral sequence that corresponds to the natural filtration of  $V$ , [13, Thm. 6.2]. Hence

$$\mu(\mathcal{T}\mathcal{F}) = \mathfrak{D}_P(1).$$

Therefore, the claim follows from [13, Thm. 6.2].  $\square$

In this paper we do not complete the fundamental cocycle as we know its counterpart as a cyclic cocycle over  $\mathcal{A}$  by the following argument. Let us recall the characteristic map

$$\chi_\varphi : C^\bullet(\mathcal{K}, V) \longrightarrow C^{\bullet+2}(\mathcal{A}) \tag{4.35}$$

for the SAYD-twisted cyclic 2-cocycle defined by the Theorem 3.15. To this end, we first prove a generalization of [3, Prop. 18]. In view of in [10],  $\mathcal{H}_n$  is realized as a bicrossed product Hopf algebra  $\mathcal{U} \bowtie \mathcal{F}^{\text{cop}}$ . Here  $\mathcal{F}$  is the commutative algebra of regular functions on the group of diffeomorphisms which preserve the origin and with identity Jacobian at the origin, and  $\mathcal{U} = U(g\ell_2^{\text{affine}})$ . The coaction involed in this bicrossed product realization is recalled below

$$\begin{aligned} \nabla : \mathcal{U} &\rightarrow \mathcal{F}^{\text{cop}} \otimes \mathcal{U}, \\ X_k &\mapsto 1 \otimes X_k + \delta_{jk}^i \otimes Y_i^j, \quad Y_i^j \mapsto 1 \otimes Y_i^j. \end{aligned} \quad (4.36)$$

In the following proposition, for any  $1 \leq j \leq m := n^2 + n$ ,

$$\nabla^j(Z) = Z_{<-j>} \otimes \cdots \otimes Z_{<-1>} \otimes Z_{<0>} \otimes 1 \otimes \cdots \otimes 1 \in \mathcal{H}_n^{\otimes m+1}. \quad (4.37)$$

**Proposition 4.4.** *The  $m := n^2 + n$ -cochain*

$$\text{TF} := (-1)^{(m-1)!} \sum_{\sigma \in S_m} (-1)^\sigma \nabla^m(Z^{\sigma(1)}) \cdots \nabla(Z^{\sigma(m)}) \in \mathcal{H}_n^{\otimes m+1} \quad (4.38)$$

*is a cyclic  $m$ -cocycle whose class  $[\text{TF}] \in HC^m(\mathcal{H}_n)$  corresponds, by the Connes-Moscovici characteristic map, to the transverse fundamental class  $[TF] \in HC^m(\mathcal{A})$ .*

*Proof.* Let  $a^i := f^i U_{\psi_i}^* \in \mathcal{A}$ , where  $0 \leq i \leq m$ ,  $f^i \in C_c^\infty(F^+ \mathbb{R}^2)$  and  $\psi \in \Gamma$ . Without loss of generality we assume that  $\psi_m \cdots \psi_0 = \text{Id}$ . The cyclic cocycle  $TF \in HC^m(\mathcal{A})$  is given by the  $m$ -cocycle

$$TF(a^0 \otimes \cdots \otimes a^m) = \int_{F^+ \mathbb{R}^n} a^0 da^1 \cdots da^m, \quad da^i = df^i U_{\psi_i}^*.$$

We note that in order to prove the claim, we need to find suitable  $h^0, \dots, h^m \in \mathcal{H}_n$  such that

$$TF(a^0 \otimes \cdots \otimes a^m) = \tau(h^0(a^0) \cdots h^m(a^m)). \quad (4.39)$$

Indeed,

$$\begin{aligned} \int_{F^+ \mathbb{R}^n} a^0 da^1 \cdots da^m &= \int_{F^+ \mathbb{R}^n} f^0 \psi_0^*(df^1) \cdots (\psi_0^* \cdots \psi_{m-1}^*)(df^m) = \\ &\int_{F^+ \mathbb{R}^n} h^0(f^0) \psi_0^*(h^1(f^1)) \cdots (\psi_0^* \cdots \psi_{m-1}^*)(h^m(f^m)) \varpi = \\ &\int_{F^+ \mathbb{R}^n} (\text{Id} \otimes \psi_0^* \otimes \cdots \otimes \psi_0^* \cdots \psi_{m-1}^*)(h^0 \otimes \cdots \otimes h^m)(f^0 \otimes \cdots \otimes f^m) \varpi, \end{aligned} \quad (4.40)$$

where the volume form on the frame bundle is

$$\varpi = \bigwedge_{i=1}^n \theta^i \wedge \bigwedge_{1 \leq i, j \leq n} \omega_j^i \quad (\text{ordered lexicographically}) \quad (4.41)$$

In the above computation we use the following notations.

$$(h^0 \otimes \dots \otimes h^m)(f^0 \otimes \dots \otimes f^m) = h^0(f^0) \dots h^m(f^m),$$

and similarly for any  $g^0, \dots, g^m \in C_c^\infty(F^+\mathbb{R}^n)$ ,

$$\begin{aligned} (\text{Id} \otimes \psi_0^* \otimes \dots \otimes \psi_0^* \dots \psi_{m-1}^*)(g^0 \otimes \dots \otimes g^m) \\ = g^0 \psi_0^*(g^1) \dots \psi_0^* \dots \psi_{m-1}^*(g^m). \end{aligned}$$

Here  $\psi^*(g)(x, y) = g(\psi(x), \psi'(x) \cdot y)$ .

For any  $f \in C_c^\infty(F^+\mathbb{R}^n)$  we have

$$df = \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial y_i^j} dy_i^j = X_i(f) \theta^i + Y_i^j(f) \omega_j^i. \quad (4.42)$$

Therefore, for  $\psi_0 \in \text{Diff}(\mathbb{R}^n)$ , and  $f^0, f^1 \in C_c^\infty(F^+\mathbb{R}^n)$  we have

$$\begin{aligned} f^0 \psi_0^*(df^1) &= f^0 \psi_0^*(X_i(f^1)) \psi_0^*(\theta^i) + f^0 \psi_0^*(Y_i^j(f^1)) \psi_0^*(\omega_j^i) \\ &= f^0 \psi_0^*(X_i(f^1)) \theta^i + f^0 \psi_0^*(Y_i^j(f^1)) (\gamma_{jk}^i(\psi_0) \theta^k + \omega_j^i) \\ &= (\text{Id} \otimes \psi_0^*)[(1 \otimes X_k + \delta_{jk}^i \otimes Y_i^j)(f^0 \otimes f^1) \theta^k + (1 \otimes Y_i^j)(f^0 \otimes f^1) \omega_j^i] \\ &= (\text{Id} \otimes \psi_0^*)((X_k)_{<-1>} \otimes (X_k)_{<0>})(f^0 \otimes f^1) \theta^k + \\ &(\text{Id} \otimes \psi_0^*)((Y_i^j)_{<-1>} \otimes (Y_i^j)_{<0>})(f^0 \otimes f^1) \omega_j^i. \end{aligned} \quad (4.43)$$

On the second equality we have used [3, (2.16)], and on the third equality we used (3.22). On the forth equality, the left coaction is (4.36).



On the other hand we have

$$\begin{aligned}
& (-1)^{(m-1)!} f^0 \psi_0^* (df^1) \psi_0^* \psi_1^* (df^2) \dots \psi_0^* \dots \psi_{m-1}^* (df^m) \\
&= \psi_0^* \dots \psi_{m-1}^* (df^m) \dots \psi_0^* \psi_1^* (df^2) \psi_0^* (df^1) f^0 \\
&= (\psi_{m-1} \dots \psi_0^* (X_i(f^m)) \theta^i + \\
&\quad \psi_{m-1} \dots \psi_0^* (Y_i^j(f^m)) (\gamma_{jk}^i (\psi_{m-1} \dots \psi_1) \theta^k + \omega_j^i)) \\
&\dots \\
&\quad \left( \psi_1 \psi_0^* (X_i(f^2)) \theta^i + \psi_1 \psi_0^* (Y_i^j(f^2)) (\gamma_{jk}^i (\psi_2 \psi_1) \theta^k + \omega_j^i) \right) \cdot \\
&\quad \left( \psi_0^* (X_i(f^1)) \theta^i + \psi_0^* (Y_i^j(f^1)) (\gamma_{jk}^i (\psi_1) \theta^k + \omega_j^i) \right) \cdot f^0 \\
&= (\text{Id} \otimes \psi_0^* \otimes \dots \otimes \psi_0^* \dots \psi_{m-1}^*) \left\{ \right. \\
&\quad \left[ ((X_i)_{<-m>} \otimes \dots \otimes (X_i)_{<0>}) \theta^i + ((Y_i^j)_{<-m>} \otimes \dots \otimes (Y_i^j)_{<0>}) \omega_j^i \right] \cdot \\
&\quad \dots \\
&\quad \left[ ((X_i)_{<-2>} \otimes (X_i)_{<-1>} \otimes (X_i)_{<0>} \otimes 1 \otimes \dots \otimes 1) \theta^i + \right. \\
&\quad \quad \left. ((Y_i^j)_{<-2>} \otimes (Y_i^j)_{<-1>} \otimes (Y_i^j)_{<0>} \otimes 1 \otimes \dots \otimes 1) \omega_j^i \right] \cdot \\
&\quad \left[ ((X_i)_{<-1>} \otimes (X_i)_{<0>} \otimes 1 \otimes \dots \otimes 1) \theta^i + \right. \\
&\quad \quad \left. ((Y_i^j)_{<-1>} \otimes (Y_i^j)_{<0>} \otimes 1 \otimes \dots \otimes 1) \omega_j^i \right] \left. \right\} (f^0 \otimes \dots \otimes f^m). \tag{4.44}
\end{aligned}$$

On the third equality, we have used the cocycle identity [10, (1.16)] in order to obtain the expressions in  $\mathcal{H}_n^{\otimes m+1}$  in the range of the coaction (4.36). We reversed the order of the multiplication in (4.44) in order to avoid obtaining elements in  $\mathcal{H}_n^{\otimes m+1}$  involving  $Y_i^j \delta_{qr}^p \in \mathcal{H}_n$  which do not belong the PBW basis of  $\mathcal{H}_n$ , [3, Prop. 3].

The coefficient of the volume form (4.41), which is an element  $H \in \mathcal{H}_n^{\otimes m+1}$ , can now be expressed by carrying out the multiplication in (4.44). Let  $(Z^1, \dots, Z^m) = (X_1, \dots, X_n, Y_1^1, \dots, Y_n^n)$ , where the right hand side is ordered lexicographically. Then

$$H = \sum_{\sigma \in S_m} (-1)^\sigma \nabla^m (Z^{\sigma(1)}) \dots \nabla (Z^{\sigma(m)}). \tag{4.45}$$

Finally, as a result of (4.40), (4.44) and (4.45) we have the element

$$\text{TF} := (-1)^{(m-1)!} \sum_{\sigma \in S_m} (-1)^\sigma \nabla^m (Z^{\sigma(1)}) \dots \nabla (Z^{\sigma(m)}) \in \mathcal{H}_n^{\otimes m+1} \tag{4.46}$$

such that for the Connes-Moscovici characteristic map (2.14) we have

$$\chi_\tau(\text{TF}) = TF \in C^m(\mathcal{A}). \quad (4.47)$$

□

Let us illustrate the proposition for  $n = 1$ . We have  $(Z^1, Z^2) = (X, Y)$ ,  $m = 1^2 + 1 = 2$  and

$$\begin{aligned} \text{TF} &= (-1)^{1!} \sum_{\sigma \in S_2} (-1)^\sigma \nabla^2(Z^{\sigma(1)}) \nabla(Z^{\sigma(2)}) \\ &= - \left( (X_{<-2>} \otimes X_{<-1>} \otimes X_{<0>}) (Y_{<-1>} \otimes Y_{<0>} \otimes 1) - \right. \\ &\quad \left. (Y_{<-2>} \otimes Y_{<-1>} \otimes Y_{<0>}) (X_{<-1>} \otimes X_{<0>} \otimes 1) \right) \\ &= -1 \otimes Y \otimes X - 1 \otimes \delta_1 Y \otimes Y - \delta_1 \otimes Y \otimes Y + 1 \otimes X \otimes Y + \delta_1 \otimes Y \otimes Y \\ &= 1 \otimes X \otimes Y - 1 \otimes Y \otimes X - 1 \otimes \delta_1 Y \otimes Y. \end{aligned} \quad (4.48)$$

Next we recall the isomorphism

$$\begin{aligned} \Psi_{\blacktriangleleft} : C^\bullet(\mathcal{H}, \mathbb{C}_\delta) &\rightarrow D^\bullet(\mathcal{U}, \mathcal{F}, \mathbb{C}_\delta) \\ \Psi_{\blacktriangleleft}(u^1 \blacktriangleleft f^1 \otimes \dots \otimes u^n \blacktriangleleft f^n) &= \\ u^1_{<-n>} f^1 \otimes \dots \otimes u^1_{<-1>} \dots u^n_{<-1>} f^n \otimes u^1_{<0>} \otimes \dots \otimes u^n_{<0>} \end{aligned} \quad (4.49)$$

defined in [10] that identifies the Hopf-cyclic complex  $C^\bullet(\mathcal{H}, \mathbb{C}_\delta)$  of a Hopf algebra  $\mathcal{H} = \mathcal{U} \blacktriangleleft \mathcal{F}$  with the diagonal subcomplex

$$D^\bullet(\mathcal{U}, \mathcal{F}, \mathbb{C}_\delta) := \mathbb{C}_\delta \otimes \mathcal{F}^{\otimes \bullet} \otimes \mathcal{U}^{\otimes \bullet}. \quad (4.50)$$

On the other hand, for  $\mathcal{U} = U(\mathfrak{g})$  we have the quasi-isomorphism

$$\mu : \mathbb{C}_\delta \otimes \mathcal{F}^{\otimes \bullet} \otimes \mathcal{U}^{\otimes \bullet} \rightarrow \mathbb{C}_\delta \otimes \mathcal{F}^{\otimes \bullet} \otimes \bigwedge^\bullet \mathfrak{g}, \quad (4.51)$$

which is the inverse of the antisymmetrization map on the level of cohomologies.

**Remark 4.5.** The transverse fundamental class  $[\text{TF}] \in HC^{n^2+n}(\mathcal{H}_n, \mathbb{C}_\delta)$  defined in (4.38) corresponds to the class

$$[1 \otimes X_1 \wedge \dots \wedge X_n \wedge Y_1^1 \wedge \dots \wedge Y_n^n], \quad (4.52)$$

in the total complex  $C^{\bullet, \bullet}(\mathfrak{g}, \mathcal{F}, \mathbb{C}_\delta)$  [10, (3.37)], by the composition of (4.49) and (4.51).

On the next move, we introduce the commutative diagram

$$\begin{array}{ccc}
C_{\mathcal{K}}^j(\mathcal{K}, V) & \xrightarrow{\bar{\chi}_\varphi} & C^{j+k}(\mathcal{H}_n, \mathbb{C}_\delta) \\
& \searrow \chi_\varphi & \downarrow T^\natural \\
& & C^{j+k}(\mathcal{A})
\end{array} \tag{4.53}$$

induced by (a decomposition of) the cup product (4.5) via a cyclic cocycle  $\varphi \in C_{\mathcal{K}}^k(\mathcal{A}, V)$  in the image of (3.30). Here  $T^\natural : \mathcal{H}_n^j \rightarrow C^j(\mathcal{A})$  is the isomorphism

$$T^\natural(h^1 \otimes \cdots \otimes h^j)(a^0 \otimes \cdots \otimes a^j) = \tau(a^0 h^1(a^1) \cdots h^j(a^j)), \tag{4.54}$$

defined in [3, (3.12)], onto the space of elementary characteristic  $j$ -cochains, [3, Section 3].

We are now ready to prove our claim. On the following proposition,  $\varphi \in C_{\mathcal{K}}^2(\mathcal{A}, V)$  is the cyclic cocycle defined in (3.66), (3.67), (3.68).

**Proposition 4.6.** *The cyclic cohomology class  $[\mathcal{T}\mathcal{F}] \in HC^4(\mathcal{K}, V)$  is mapped by  $\chi_\varphi : HC^4(\mathcal{K}, V) \rightarrow HC^6(\mathcal{A}_\Gamma)$  to the transverse characteristic class  $[TF] \in HC^6(\mathcal{A}_\Gamma)$ .*

*Proof.* By the diagram (4.53) we understand that it is enough to observe  $[\bar{\chi}_\varphi(\mathcal{T}\mathcal{F})] = [TF] \in HC^6(\mathcal{H}_n)$ . This, in turn, follows from the observation

$$\mu \circ \psi_{\blacktriangleright}([\bar{\chi}_\varphi(\mathcal{T}\mathcal{F})]) = \mu \circ \psi_{\blacktriangleright}([TF]) = [1 \otimes X_1 \wedge X_2 \wedge Y_1^1 \wedge \cdots \wedge Y_2^2], \tag{4.55}$$

thanks to the large kernel of (4.51). Hence the result follows since  $\mu \circ \psi_{\blacktriangleright}$  is an isomorphism on the level of cohomologies.  $\square$

In the following we present the image of the cyclic cocycles  $\mathcal{G}\mathcal{V}$ ,  $\mathcal{R}_1$ ,  $\mathcal{R}_2$ ,  $\mathcal{R}_3$ ,  $\mathcal{R}_4$  under the characteristic map  $\chi_\varphi : C^\bullet(\mathcal{K}, V) \rightarrow C^{\bullet+2}(\mathcal{A})$ . These are cyclic cocycles in  $C^\bullet(\mathcal{A})$ . We do not display the detailed account of the computation as it is lengthy and straightforward.

$$\begin{aligned}
\chi_\varphi(\mathcal{G}\mathcal{V})(a_0 \otimes \cdots \otimes a_5) &= \sum_{k=1}^3 \sum_{1 \leq i, j \leq 2} \sum_{\sigma, \gamma, \eta \in S_2} 2 \cdot (-1)^\sigma (-1)^\gamma (-1)^{k-1} \\
&\left\{ -\tau \left( a_0 \delta_{i\eta(1)}^i(a_1) \delta_{j\eta(2)}^j(a_2) Y_{\mu^k(\sigma(1))}^{\mu^k(\sigma(2))}(a_3) Y_{\mu^k(\gamma(1))}^{\mu^k(\sigma(1))}(a_4) Y_{\mu^k(\gamma(2))}^{\mu^k(\sigma(1))}(a_5) \right) \right. \\
&+ \tau \left( a_0 \delta_{i\eta(1)}^i(a_1) Y_{\mu^k(\sigma(1))}^{\mu^k(\sigma(2))}(a_2) \delta_{j\eta(2)}^j(a_3) Y_{\mu^k(\gamma(1))}^{\mu^k(\sigma(1))}(a_4) Y_{\mu^k(\gamma(2))}^{\mu^k(\sigma(1))}(a_5) \right) \\
&- \tau \left( a_0 \delta_{i\eta(1)}^i(a_1) Y_{\mu^k(\sigma(1))}^{\mu^k(\sigma(2))}(a_2) Y_{\mu^k(\gamma(1))}^{\mu^k(\sigma(1))}(a_3) \delta_{j\eta(2)}^j(a_4) Y_{\mu^k(\gamma(2))}^{\mu^k(\sigma(1))}(a_5) \right) \\
&+ \tau \left( a_0 \delta_{i\eta(1)}^i(a_1) Y_{\mu^k(\sigma(1))}^{\mu^k(\sigma(2))}(a_2) Y_{\mu^k(\gamma(1))}^{\mu^k(\sigma(1))}(a_3) Y_{\mu^k(\gamma(2))}^{\mu^k(\sigma(1))}(a_4) \delta_{j\eta(2)}^j(a_5) \right) \\
&- \tau \left( a_0 Y_{\mu^k(\sigma(1))}^{\mu^k(\sigma(2))}(a_1) \delta_{i\eta(1)}^i(a_2) \delta_{j\eta(2)}^j(a_3) Y_{\mu^k(\gamma(1))}^{\mu^k(\sigma(1))}(a_4) Y_{\mu^k(\gamma(2))}^{\mu^k(\sigma(1))}(a_5) \right) \\
&+ \tau \left( a_0 Y_{\mu^k(\sigma(1))}^{\mu^k(\sigma(2))}(a_1) \delta_{i\eta(1)}^i(a_2) Y_{\mu^k(\gamma(1))}^{\mu^k(\sigma(1))}(a_3) \delta_{j\eta(2)}^j(a_4) Y_{\mu^k(\gamma(2))}^{\mu^k(\sigma(1))}(a_5) \right) \\
&- \tau \left( a_0 Y_{\mu^k(\sigma(1))}^{\mu^k(\sigma(2))}(a_1) \delta_{i\eta(1)}^i(a_2) Y_{\mu^k(\gamma(1))}^{\mu^k(\sigma(1))}(a_3) Y_{\mu^k(\gamma(2))}^{\mu^k(\sigma(1))}(a_4) \delta_{j\eta(2)}^j(a_5) \right) \\
&- \tau \left( a_0 Y_{\mu^k(\sigma(1))}^{\mu^k(\sigma(2))}(a_1) Y_{\mu^k(\gamma(1))}^{\mu^k(\sigma(1))}(a_2) \delta_{i\eta(1)}^i(a_3) \delta_{j\eta(2)}^j(a_4) Y_{\mu^k(\gamma(2))}^{\mu^k(\sigma(1))}(a_5) \right) \\
&+ \tau \left( a_0 Y_{\mu^k(\sigma(1))}^{\mu^k(\sigma(2))}(a_1) Y_{\mu^k(\gamma(1))}^{\mu^k(\sigma(1))}(a_2) \delta_{i\eta(1)}^i(a_3) Y_{\mu^k(\gamma(2))}^{\mu^k(\sigma(1))}(a_4) \delta_{j\eta(2)}^j(a_5) \right) \\
&\left. - \tau \left( a_0 Y_{\mu^k(\sigma(1))}^{\mu^k(\sigma(2))}(a_1) Y_{\mu^k(\gamma(1))}^{\mu^k(\sigma(1))}(a_2) Y_{\mu^k(\gamma(2))}^{\mu^k(\sigma(1))}(a_3) \delta_{i\eta(1)}^i(a_4) \delta_{j\eta(2)}^j(a_5) \right) \right\}.
\end{aligned} \tag{4.56}$$

$$\begin{aligned}
&\chi_\varphi(\mathcal{R}_2)(a_0 \otimes \cdots \otimes a_3) \\
&= \sum_{\sigma \in S_2} (-1)^\sigma \left\{ -\tau(a_0 \delta_{2\sigma(1)}^1(a_1) \delta_{1\sigma(2)}^2(a_2) Y_2^2(a_3)) \right. \\
&- \tau(a_0 \delta_{1\sigma(1)}^2(a_1) \delta_{2\sigma(2)}^1(a_2) Y_2^2(a_3)) + \tau(a_0 \delta_{2\sigma(1)}^1(a_1) Y_2^2(a_2) \delta_{1\sigma(2)}^2(a_3)) \\
&+ \tau(a_0 \delta_{1\sigma(1)}^2(a_1) Y_2^2(a_2) \delta_{2\sigma(2)}^1(a_3)) - \tau(a_0 Y_2^2(a_1) \delta_{2\sigma(1)}^1(a_2) \delta_{1\sigma(2)}^2(a_3)) \\
&\left. - \tau(a_0 Y_2^2(a_1) \delta_{1\sigma(1)}^2(a_2) \delta_{2\sigma(2)}^1(a_3)) \right\}.
\end{aligned} \tag{4.57}$$

$$\chi_\varphi(\mathcal{R}_3)(a_0 \otimes a_1 \otimes a_2) = \sum_{1 \leq i, j \leq 2} \sum_{\sigma \in S_2} 2 \cdot (-1)^\sigma \tau(a_0 \delta_{i\sigma(1)}^i(a_1) \delta_{j\sigma(2)}^j(a_2)). \tag{4.58}$$

[illegible]

Finally,

$$\begin{aligned}
\chi_\varphi(\mathcal{R}_4)(a_0 \otimes a_1 \otimes a_2) &= \varphi(c_2 \otimes a_0 \otimes a_1 \otimes a_2) \\
&= \sum_{\sigma \in S_2} (-1)^\sigma \left\{ \tau(a_0 \delta_{2\sigma(1)}^1(a_1) \delta_{1\sigma(2)}^2(a_2)) + \tau(a_0 \delta_{1\sigma(1)}^2(a_1) \delta_{2\sigma(2)}^1(a_2)) \right\}.
\end{aligned} \tag{4.60}$$

**Remark 4.7.** One knows that the characteristic map  $\chi_\tau : C^\bullet(\mathcal{H}, \mathbb{C}_\delta) \rightarrow C^\bullet(\mathcal{A})$  is injective [2]. Since  $\chi_\varphi(\mathcal{TF})$ ,  $\chi_\varphi(\mathcal{GV})$ ,  $\chi_\varphi(\mathcal{R}_1)$ ,  $\chi_\varphi(\mathcal{R}_2)$ ,  $\chi_\varphi(\mathcal{R}_3)$ , and  $\chi_\varphi(\mathcal{R}_4)$ , are all in the range of  $\chi_\tau$ , as a byproduct of our study in this paper, one calculates cyclic cocycles representing a basis for  $HP^\bullet(\mathcal{H}_2, \mathbb{C}_\delta)$ .

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